Notes on Electromagnetic Waves in a Plasma

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These notes outline some of the key ideas concerning electromagnetic waves in a plasma. The subjects of plasma physics and plasma astrophysics are vast. The notes here are necessarily superficial. A more complete introduction can be found in *An Introduction to Plasma Theory* by Dwight Nicholson. Yet more detailed treatments can be found in *Radiation Processes in Plasmas* by G. Bekefi and *Plasma Astrophysics* by D. Melrose. *The Magnetoionic Theory* is treated at length in the classic monograph by J. Ratcliffe.

1 Length scales and time scales

A plasma is a ionized gas composed of roughly equal numbers of electrons and ions (quasi-neutral). To keep things simple, we will be considering a fully ionized gas. Furthermore, we will only consider a hydrogen plasma; that is, a gas of equal numbers of electrons and protons. A plasma is characterized by collective effects.

1.1 Debye shielding

Consider a test particle with a positive charge q_T. In a vacuum it would have an electric potential

$$\phi = \frac{q_T}{r}.$$

If we now introduce the test charge into a uniform plasma of infinite extent it repels surrounding protons and attracts the electrons. Collectively, they form a shielding cloud of particles. If we wait until the system equilibrates, we find that the electric potential due to the test charge in the plasma is

$$\phi = \frac{q_T}{r} e^{-r/\lambda_D}.$$

The potential of the test charge falls off much more rapidly in a plasma than it does in a vacuum due to the shielding influence of the charge cloud. For distances $r \gg \lambda_D$, the *Debye length*, the influence of the test charge is negligible. This phenomenon is called *Debye shielding*. The Debye length can be shown to be

$$\lambda_D = \left(\frac{k_B T}{4\pi n e^2}\right)^{1/2},$$

where T is the temperature (taken here to be the same for electrons and protons), n is the electron (or proton) number density, and e is the electric charge. Numerically, $\lambda_D \approx 5\sqrt{T/n}$. In the solar wind, $T \approx 10^5$ K and $n \sim 5$ cm^{-3}, yielding $\lambda_D \sim 7$ m. A defining characterisitic of a plasma is that it has a large number of particles in Debye sphere: i.e., $\Lambda = (4\pi/3)n\lambda_D^3 \gg 1$, a number sometimes refered to as the *plasma parameter*.

1.2 Plasma frequency

Consider as uniform slab of plasma of thickness L in the x direction and very large dimensions in the y and z dimensions. The plasma is composed of equal numbers of electrons and protons. We take the proton mass to be effectively infinite compare to the electron mass. The protons are therefore effectively fixed in place. Suppose we displaced the electrons from the protons by a distance $\delta \ll L$. An electric field is set up that would exert a force on the electrons, pulling them back to the protons. Letting the electrons go, they would rush back toward the protons, overshoot, and an oscillation would be set up with a characteristic frequency.

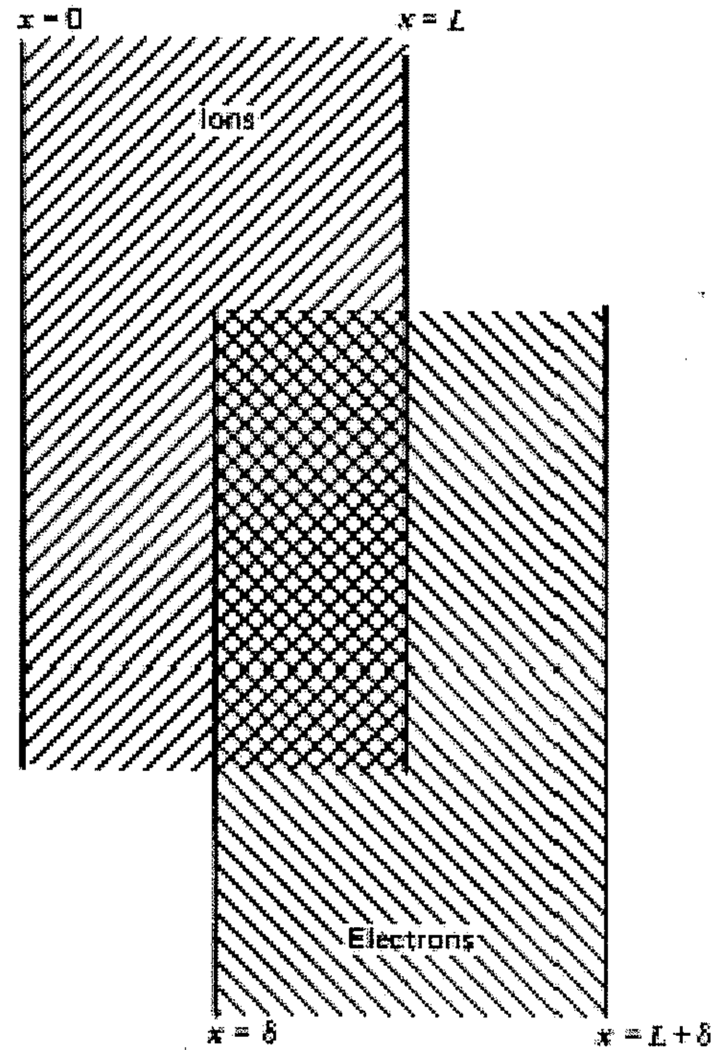
Use Poisson's equation in one dimension: $\partial_x E = 4\pi\rho$. With the displacement δ we then have $E = 4\pi n e\delta$. The charge per unit area is just $-neL$ and the (force per unit area) is (the electric field) times (the charge per unit area): $F = m_e\ddot{\delta} = (-neL)(4\pi ne\delta) = -4\pi n^2 e^2 \delta L$. But the (force per unit area) is just the (mass per unit area) times acceleration. The (mass per unit area) is nM_eL and so we can write

$$n m_e L\ddot{\delta} = 4\pi n^2 e^2 L\delta \longrightarrow \ddot{\delta} + \frac{4\pi n e^2}{m_e}\delta = 0$$

which is the equation of a harmonic oscillator with a frequency $\omega_{pe} \equiv (4\pi n e^2/m_e)^{1/2}$. This is the *electron plasma frequency*. A similar frequency can be defined for the protons (or more generally, ions) that is smaller then ω_{pe} in the ratio $(m_e/m_i)^{1/2}$. Note that in a thermal plasma, if we take the product of the Debye length and the plasma frequency, we obtain

$$\lambda_D\omega_{pe} = \frac{(k_BT/4\pi n e^2)^{1/2}}{(4\pi n e^2/m_e)^{1/2}} = \sqrt{\frac{k_BT}{m_e}} = v_{th}.$$

The cyclic plasma frequency is just $\nu_{pe} = \sqrt{ne^2/\pi m_e} \approx 9n_e^{1/2}$ kHz.


x = 0
x = L
Ions
Electrons
x = δ
x = L + δ

1.3 Gyrofrequency

Now consider the motion of electrons and protons in the presence of a magnetic field. The Lorentz force is

$$\mathbf{F} = m\dot{\mathbf{v}} = \frac{q}{c}\mathbf{v} \times \mathbf{B}.$$

For simplicity, we take $\mathbf{B} = B_o\hat{\mathbf{z}}$. We can then write three equations of (electron) motion:

$$\dot{v}_x = -\omega_{Be}v_y \qquad \dot{v}_y = \omega_{Be}v_x \qquad \dot{v}_z = 0$$

where $\omega_{Be} = eB/m_ec$ is the *electron gyrofrequency* and ϕ is an arbitrary phase. We then have

$$v_x = v_\perp \cos(\omega_{Be}t + \phi) \qquad \dot{v}_y = v_\perp \sin(\omega_{Be}t + \phi)$$

and $v_z = v_\parallel$ =constant and $v_\perp = (v_x^2 + v_y^2)^{1/2}$ =constant.
Integration yields

$$v_x = x_g + \frac{v_\perp}{\omega_{Be}} \cos(\omega_{Be}t + \phi) \qquad v_y = v_g + \frac{v_\perp}{\omega_{Be}} \sin(\omega_{Be}t + \phi)$$

The electron's trajectory is a helix, with uniform speed parallel to the magnetic field and circular motion perpendicular to the magnetic field. The radius of the circle is the *Larmor radius* $r_L = v_\perp/\omega_{Be}$. Taking $v_\perp \sim v_{th}$ we have in analogy to the product $\lambda_D\omega_p$ that $r_L\omega_{Be} = v_{th}$. The cyclic gyrofrequency is $\Omega_{Be} = eB/2\pi m_ec \approx 2.8B$ MHz, with B in Gauss.

1.4 Collision frequency

The derivation of the collision frequency is a plasma is somewhat involved and therefore won't be reproduced here. It is interesting to note that if the collision time is defined to be the time required for a given particle to deviated from its initial trajectory by 90°, the cumulative effects of small-angle collisions far outweight the occasional large-angle collision. If one does the calculation, it is found that the electron-proton collision frequency for a large-angle collision is of order

$$\nu_L \approx \frac{4\pi ne^4}{m_e^2v^3}$$

where v is the electron speed. In contrast, the collision rate due to the cumulative effects of small angle collisions is

$$\nu_c = \frac{8\pi ne^4 \ln\Lambda}{m_e^2v^3}.$$

We see that the electron-proton collision frequency ν_c due to small angle collisions is a factor $2 \ln \Lambda$ larger than ν_L and the collision time is therefore correspondingly shorter. Note the reappearance of the plasma parameter. Note also that the collision frequency in both cases is proportional to $1/v^3$. The kinetic energy of the particle is $E = mv^2/2$ and so $\nu_c \propto E^{-3/2}$. Calculations show that the electron-electron collision frequency is approximately the same as the electron-proton collision frequency. The proton-proton collision frequency is found to be a factor $\approx (m_p/m_e)^{1/2}$ smaller than the electron-proton, and the proton-electron collision frequency is $\approx (m_p/m_e)$ smaller.

2 EM waves in a plasma

In this section we'll consider EM waves in a vacuum, an unmagnetized plasma, and in a magnetized plasma.

2.1 EM waves in a vacuum

Recall Maxwell's equations:

$$\begin{aligned} \nabla \cdot \mathbf{D} &= 4\pi\rho \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{H} &= \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c}\mathbf{J} \end{aligned}$$

Here $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$, where ϵ is the dielectric constant and μ is the magnetic permeability. In a vacuum, we have no charges ρ or currents $\mathbf{J}$, and $\mu = \epsilon = 1$). Maxwell's equations then become

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} &= \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \end{aligned}$$

Taking the curl of the third equation and the partial derivative of the last equation wrt time yields

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{1}{c}\nabla \times \frac{\partial \mathbf{B}}{\partial t}$$

and

$$\nabla \times \frac{\partial \mathbf{B}}{\partial t} = \frac{1}{c}\frac{\partial^2 \mathbf{E}}{\partial t^2},$$

respectively, and so

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Using the identity $\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$ we end with

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0.$$

This is a (vector) wave equation for the electric field with plane wave solutions that go as $e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$. Substituting this into the wave equation yields $-\omega^2\mathbf{E} + c^2k^2\mathbf{E}$, or $k^2 = \omega^2/c^2$. This last relation, relating ω and k, is called the *dispersion relation*. Before continuing, it is worth taking a brief detour to consider a plasma as a dielectric.

2.2 Plasma as a dielectric

The application of an electric field to a dielectric induces a polarization $\mathbf{P}$, defined to be the induced dipole moment per unit volume

$$\mathbf{P} \equiv \int d\mathbf{r}\rho(\mathbf{r})\mathbf{r}.$$

For the case of a uniform plasma, if n electrons per unit volume are moved a distance $\mathbf{r}$, the equivalent dipole moment per unit volume is just $\mathbf{P}= ne\mathbf{r}$.

By definition, $\mathbf{D}$=$\mathbf{E}$+$4\pi\mathbf{P}$. If $\mathbf{P}$ is a linear function of the electric field $\mathbf{E}$ such that $\mathbf{P}$=$\chi\mathbf{E}$, we can write $\mathbf{D}= \epsilon\mathbf{E}$, where $\epsilon = 1 + 4\pi\chi$ is the *dielectric constant* of the medium.

Now consider the response of a (unmagnetized) plasma to an electromagnetic wave with an electric field $\mathbf{E} \sim \exp(i\omega t)$. We ignore the motion of the massive protons compared to that of the electrons. We also ignore collisions. The equation of motion of a single electron is then

$$m\ddot{\mathbf{r}} = e\mathbf{E} = -m\omega^2\mathbf{r}$$

so that $\mathbf{r} = (-e/m\omega^2)\mathbf{E}$. From this and the expression for $\mathbf{P}$ above we find that $\mathbf{P} = (-ne^2/m\omega^2)\mathbf{E}$, or $\chi = -ne^2/m\omega^2$. It then follows that for an unmagetized plasma, the dielectric constant is

$$\epsilon = 1 - \frac{4\pi ne^2}{m\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega^2}.$$

The response of the plasma to an electromagnetic wave is embodied in the dielectric constant.

2.3 EM waves in an unmagnetized plasma

Continuing, we consider Maxwell's equations for an unmagnetized (cold) plasma:

$$\nabla \cdot \mathbf{D} = 0$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$$

Noting that $\mathbf{D} = \epsilon \mathbf{E}$ and proceeding as before, we obtain

$$\nabla^2 \mathbf{E} + \frac{\epsilon}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0,$$

Substituting a plane wave solution, we obtain the following dispersion relation $k^2 = \epsilon\omega^2/c^2$, or

$$k^2 = \frac{\omega^2}{c^2}\Big(1 - \frac{\omega_{pe}^2}{\omega^2}\Big) \longrightarrow \omega^2 = \omega_{pe}^2 + k^2c^2$$

We note several features of EM waves in a plasma. First, $\epsilon = k^2c^2/\omega^2 \equiv \mu^2$, where μ is the *refractive index*. The group velocity is $v_{gr} = \partial\omega/\partial k = \mu c$ and the phase velocity is $v_{ph} = \omega/k = c/\mu$. Since $\mu < 1$ in a plasma, $v_{gr} < c$ and $v_{ph} > c$. Second, note that if $\omega < omega_{pe}$ we have $\epsilon < 0$ and μ is imaginary. In other words, EM wave are non-propagating when their frequency is less than the plasma frequency.

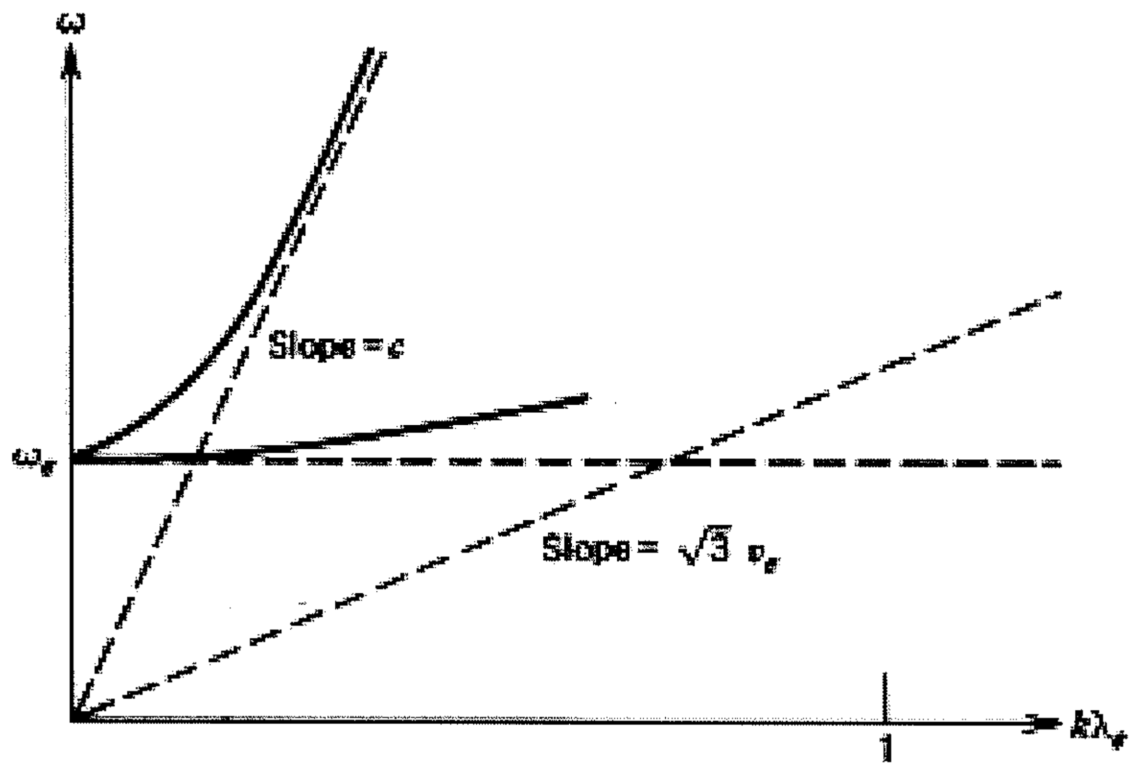
2.4 EM waves in a magnetized plasma

A magnetized plasma is a very complex and rich medium. It can support *many* wave modes; not just EM modes, but a large variety of MHD modes as well. Here we touch on just two special cases to illustrate the properties of EM waves in a magnetized plasma. I will comment on the slightly more general framework provided by the magnetoionic theory in the last section. In all cases, we will only consider electron motion, taking the ions to be fixed; i.e., a cold plasma.

2.4.1 Case 1: Wave propagation perpendicular to the magnetic field

As before, consider a plasma permeated by a constant magnetic field $\mathbf{B} = B_\circ\hat{\mathbf{z}}$. In general, for an EM perturbation, the force on an electron is given by the Lorentz force and the electron equation of motion is:

$$F = m_e n\dot{\mathbf{v}} = -en\mathbf{E} - \frac{e}{c}n\mathbf{v} \times \mathbf{B}.$$


ω
Slope = c
ω_e
Slope = $\sqrt{3}\ v_e$
1
$k\lambda_e$

We consider EM waves propagating perpendicular to $\mathbf{B}$. There are two possibilities. The electric field can be parallel to $\mathbf{B}$, or the the electric field can be in the $x-y$ plane perpendicular to $\mathbf{B}$. In the first case, the electric field induces electron motion parallel to $\mathbf{B}$. In this case, the $v \times B$ term vanishes and the problem becomes identical to that in an unmagnetized plasma. The dispersion equation for this *ordinary mode* is therefore the same as that for and EM wave in an unmagnetized plasma: $\omega^2 = \omega_{pe}^2 + k^2c^2$.

Now consider the second possibility. The electric field induces motion in the $x-y$ plane and the $\mathbf{v} \times \mathbf{B}$ force then causes another velocity component in the $x-y$ plane. Note that the electron motion of the ordinary mode wave and this *extraordinary mode* wave are orthogonal. Again assuming plane wave solutions we can write

$$-i\omega m_e v_x = -eE_x - \frac{e}{c}v_y B_\circ \qquad -i\omega m_e v_y = -eE_y + \frac{e}{c}v_x B_\circ,$$

and from Maxwell's equations for $\nabla \times \mathbf{E}$ and $\nabla \times \mathbf{B}$ we find that

$$v_x = \frac{-i\omega}{4\pi ne}E_x \qquad v_y = \Big(\frac{ik^2c^2}{4\pi ne\omega} + \frac{-i\omega}{4\pi ne}\Big)E_y.$$

Substituting into the expressions above, we can write in matrix notation

$$\begin{bmatrix} -i\omega m_e\left(\frac{-i\omega}{4\pi ne}\right) + e & \frac{e}{c}B_\circ\left(\frac{ik^2c^2}{4\pi ne\omega} + \frac{-i\omega}{4\pi ne}\right) \\ -\frac{eB_\circ}{c}\left(\frac{-i\omega}{4\pi ne}\right) & -i\omega m\left(\frac{ik^2c^2}{4\pi ne\omega} + \frac{-i\omega}{4\pi ne} + e\right) \end{bmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

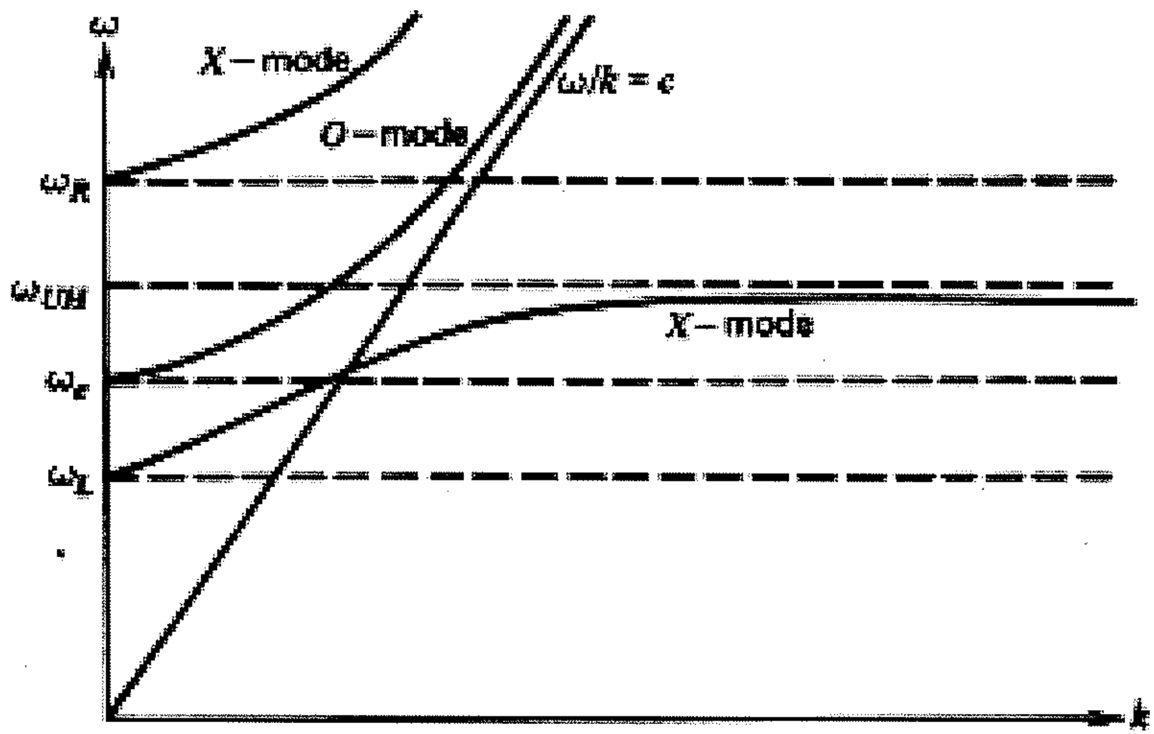
The determinant of the coefficients must be zero, and so we find after some algebra the dispersion relation for the extraordinary mode:

$$\Big(1 - \frac{\omega^2}{\omega_{pe}^2}\Big)\Big(1 + \frac{k^2c^2}{\omega_{pe}^2} - \frac{\omega^2}{\omega_{pe}^2}\Big) + \frac{\omega_{Be}^2 k^2c^2}{\omega_{pe}^2} - \frac{\omega^2\omega_{Be}^2}{\omega_{pe}^2} = 0$$

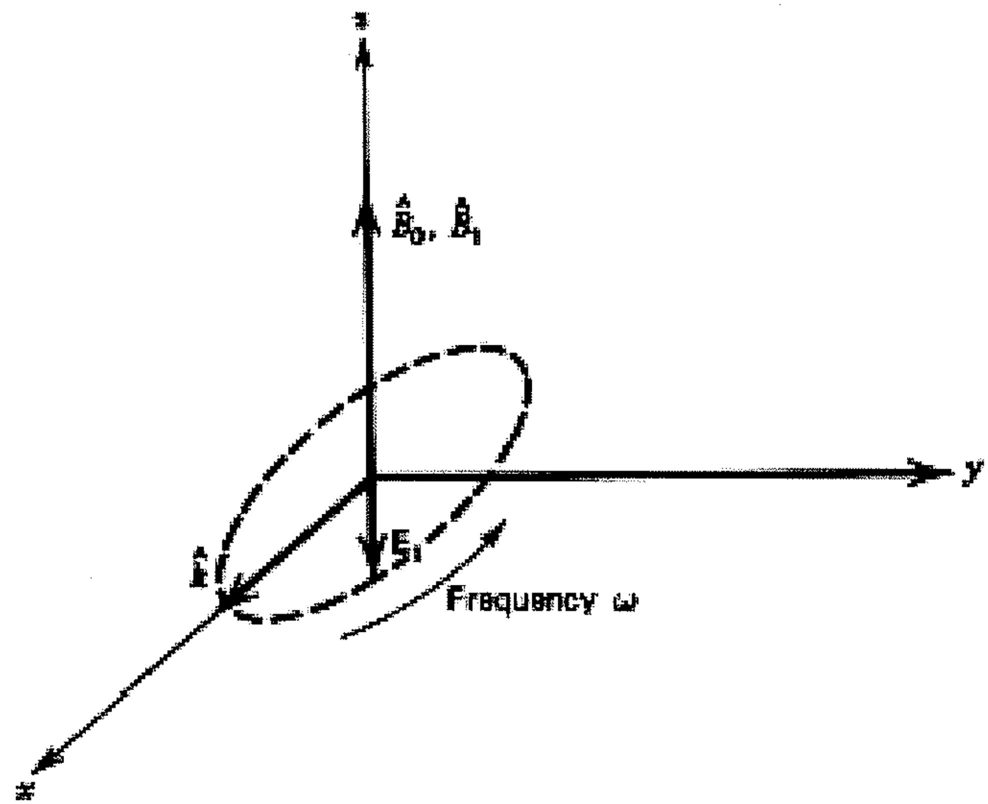
From this cumbersome expression we can extract a factor $k^2c^2/\omega^2 = \mu_\nu^2$ and, again after some manipulation, we obtain

$$\mu^2 = \frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega^2}\frac{\omega^2 - \omega_{pe}^2}{\omega^2 - \omega_{UH}^2}.$$

Note that a new frequency has been introduced: $\omega_{UH}^2 = \omega_{pe}^2 + \omega_{Be}^2$ defines the *upper hybrid frequency.* The x-mode dispersion relation is complex. Two important properties are the presence of *resonances* and *cutoffs.* A resonance is occurs at any frequency where $k \to \pm\infty$; a cutoff occurs at any frequency where $k \to 0$. A casual look at the dispersion relation shows that the resonances occur at $\omega = 0$ and $\omega = \omega_{UH}$. The cutoffs are found by setting $k = 0$ and then solving for ω. One finds that



ω
X − mode
ω/k = c
O − mode
ω_R
ω_UH
X − mode
ω_p
ω_L
k


y
Frequency ω

$$\omega = \left[\omega_{pe}^2 + \frac{\omega_{Be}^2}{2} \pm \omega_{Be}\sqrt{\omega_{pe}^2 + \omega_{Be}^2/4}\right]^{1/2}$$

The x-mode is somewhat strange in that it is partially transvers and partially longitudinal. That is, the electric field of the EM mode has components both parallel and perpendendicular to the direction of propagation. The electric vector traces out an ellipse in the $x-y$ plane. Note that in the high frequency limit, the o- and x-modes become fully linearly polarized.

2.4.2 Case 2: Wave propagation parallel to the magnetic field

We now consider waves propagating parallel to $\mathbf{B}$. Performing an analysis similar to Case 1, we find that

$$-ikB_y = \frac{-4\pi ne}{c}v_x - \frac{i\omega}{c}E_x \qquad ikB_x = \frac{-4\pi ne}{c}v_y - \frac{i\omega}{c}E_y$$

and

$$v_x = -\frac{c}{4\pi n}\left(\frac{-ik^2c}{\omega} + \frac{i\omega}{c}\right)E_x \qquad v_y = -\frac{c}{4\pi n}\left(\frac{-ik^2c}{\omega} + \frac{i\omega}{c}\right)E_y$$

which can be used to form a matrix equation, the solution of which yields

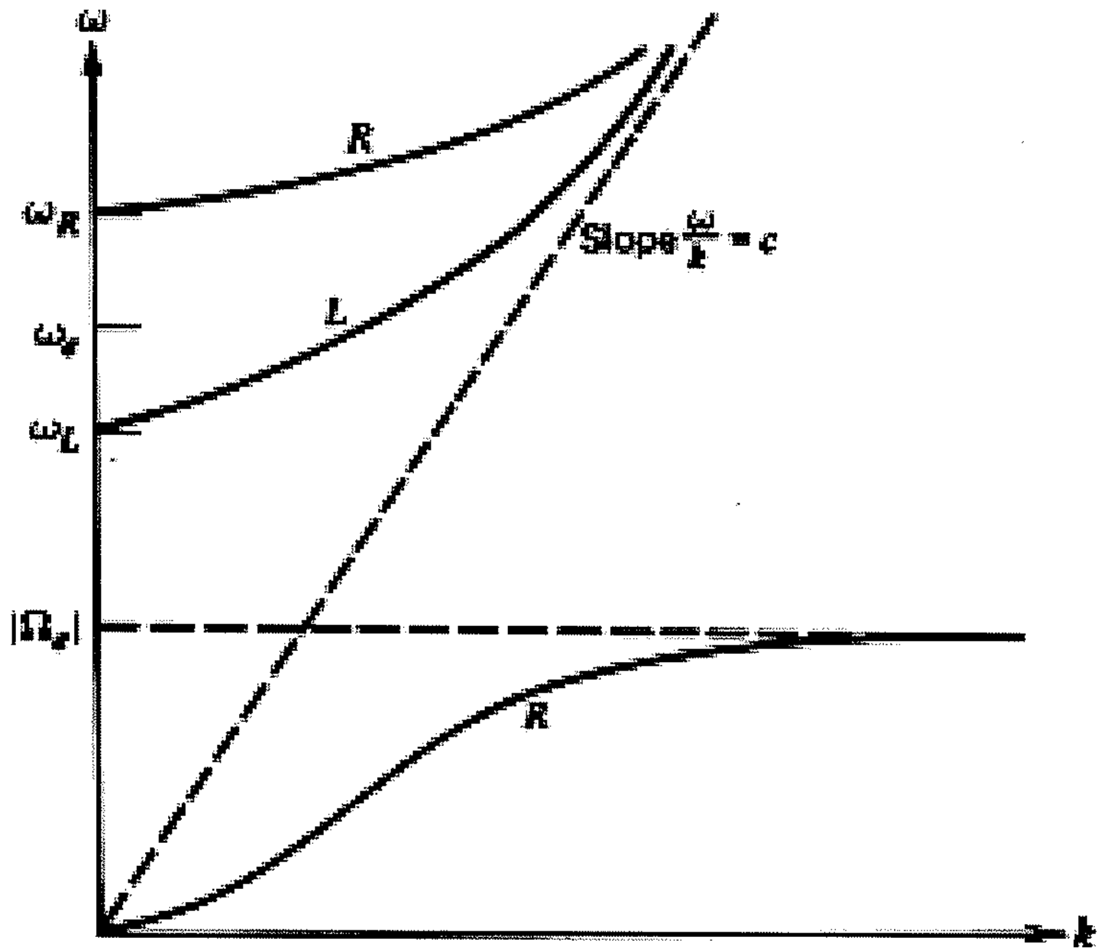
$$\left(1 + \frac{k^2c^2}{\omega_{pe}^2} - \frac{\omega^2}{\omega_{pe}^2}\right)^2 = \frac{\omega_{Be}^2}{\omega_{pe}^2}\left(\omega - \frac{k^2c^2}{\omega}\right)^2$$

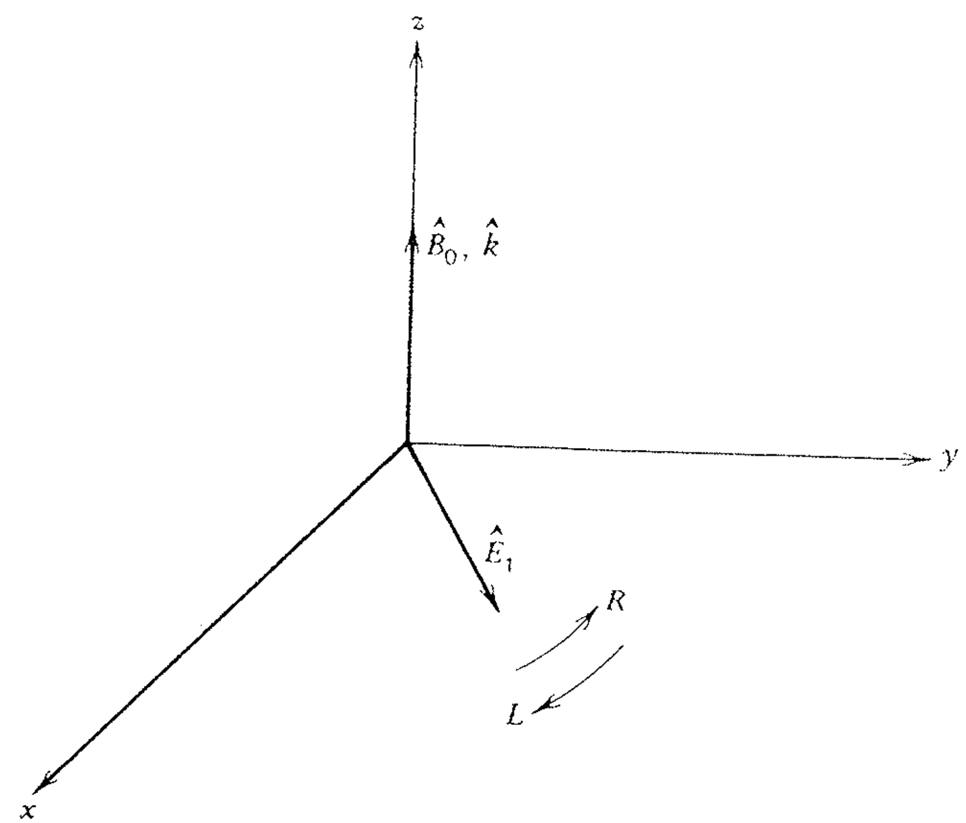
Taking the square root of the above and again extracting the refractive index yields the follwoing dispersion relations:

$$\mu^2 = \frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_{pe}^2/\omega^2}{1 \pm \omega_{Be}/\omega}$$

There are two wave modes corresponding to the two signs in the denominator of the last term. Taking the "+" sign, the corresponding wave mode is *right circularly polarized* (R-wave) and for the "-" sign, it is *left circularly polarized* (L-wave). That is, the electric field vector of the EM mode traces out circle in the counter-clockwise (RCP) or clockwise (LCP) sense. There are again curoffs and resonances in the dispersion relations. The R-wave has a resonance at $\omega = -\omega_{Be}$. This makes physical sense because the sense and frequency of rotation of the electric field matches the gyromotion of the electrons. The L-wave has no resonances. The cutoffs are again found by setting $k = 0$:

$$\omega_{R,L} = \mp\frac{\omega_{Be}}{2} + \sqrt{\omega_{pe}^2 + \omega_{Be}^2/4}$$


ω
R
ω_R
Slope ω/k = c
L
ω_p
ω_L
|Ω_e|
R
k


z
$\hat{B}_0$, $\hat{k}$
y
$\hat{E}_1$
R
L
x

It is possible to show that these cutoffs are equivalent to those found for the x-mode in the case 1. Note again that in the high-frequency limit, the R- and L-mode propagate near c.

2.5 Faraday rotation

An important phenomenon arising from the fact that the x- and o-modes are described by different dispersion relations is *Faraday rotation*. A *linearly* polarlized wave can be regarded as a superposition of the two circularly polarized modes. In a magnetoactive medium, the propagation speeds of the two modes differ and the two circularly polarized modes therefor become increasingly out of phase as they propagate. As a result, the plane of polarization rotates as the wave propagates.

The change in the phase ϕ of a circularly polarized mode as it travels a distance d is just $2\pi d/\lambda = \mathbf{k}\cdot\mathbf{d}$. If the properties of the medium change along d we then have

$$\phi_{R,L} = \int_0^d k_{R,L} ds$$

where $k_{R,L} = \omega\sqrt{\epsilon_{R,L}}/c$. If $X \ll 1$ and $Y \ll 1$ we find (solving for k using the result from §2.4.2) that

$$k_{R,L} \approx \frac{\omega}{c}\Big[1 - \frac{\omega_{pe}^2}{2\omega^2}\Big(1 \mp \frac{\omega_{Be}}{\omega}\Big)\Big].$$

The angle through which the plane of polarization rotates is then

$$\Delta\theta = \frac{1}{2}\int_0^d (k_R - k_L) ds = \frac{1}{2}\int_0^d \frac{\omega_{pe}^2\omega_{Be}}{c\omega^2} ds = \frac{2\pi e^3}{m_e^2 c^2 \omega^2}\int_0^d n B_{\|} ds.$$

Faraday rotation is an observable effect and is an important diagnostic of cosmic magnetic fields. Note, however, that the integral over $B_{\|}$ is constrained. While we used the results of our analysis for propagation parallel to $\mathbf{B}$, a more general treatment yields the above result.

2.6 Plasma waves

We introduced the electron plasma frequency in §1 as a natural oscillation frequency of a plasma. Plasma oscillations are electrostatic oscillations, not an EM mode. Nevertheless, they are worth mentioning for later use. Their dispersion relation can be simply written as $\omega^2 = \omega_{pe}^2$. Since there is no k dependence, plasma waves (or Langmuir waves) are non-dispersive and therefore non-propagating. However, if we include consideration of the plasma temperature, a longitudinal disturbance causes adiabatic compression and a pressure term comes in as $\nabla P_e = 3k_B T \nabla n$. The dispersion relation for plasma waves is then modified as

$$\omega^2 = \omega_{pe}^2 + 3k^2 v_t h^2 \longrightarrow \omega \approx \pm\omega_{pe}(1 + 3k^2\lambda_D^2/2)$$

The plasma waves are thus dispersive and have a group velocity $v_{gr} \approx 3(k\lambda_D)v_{th}$.

3 Magnetionic theory

The magnetoionic theory is a generalization of the above special cases. That is, it considers modes that propagate in arbitrary directions relative to the magnetic field in a cold plasma wherein the ions are fixed. It was originally developed to describe the propagation of EM modes in the ionosphere but it is valid under many circumstances in the solar corona as well.

Two approximate regimes are commonly considered in the magnetoionic theory that correspond rather closely to Case 1 and Case 2 considered in the previous section. Let θ be the angle between $\mathbf{k}$ and $\mathbf{B}$. Defining the two magnetoionic parameters $X = \omega_{pe}^2/\omega^2$ and $Y = \omega_{Be}/\omega$ one can define two propagation regimes, depending on whether the quantity

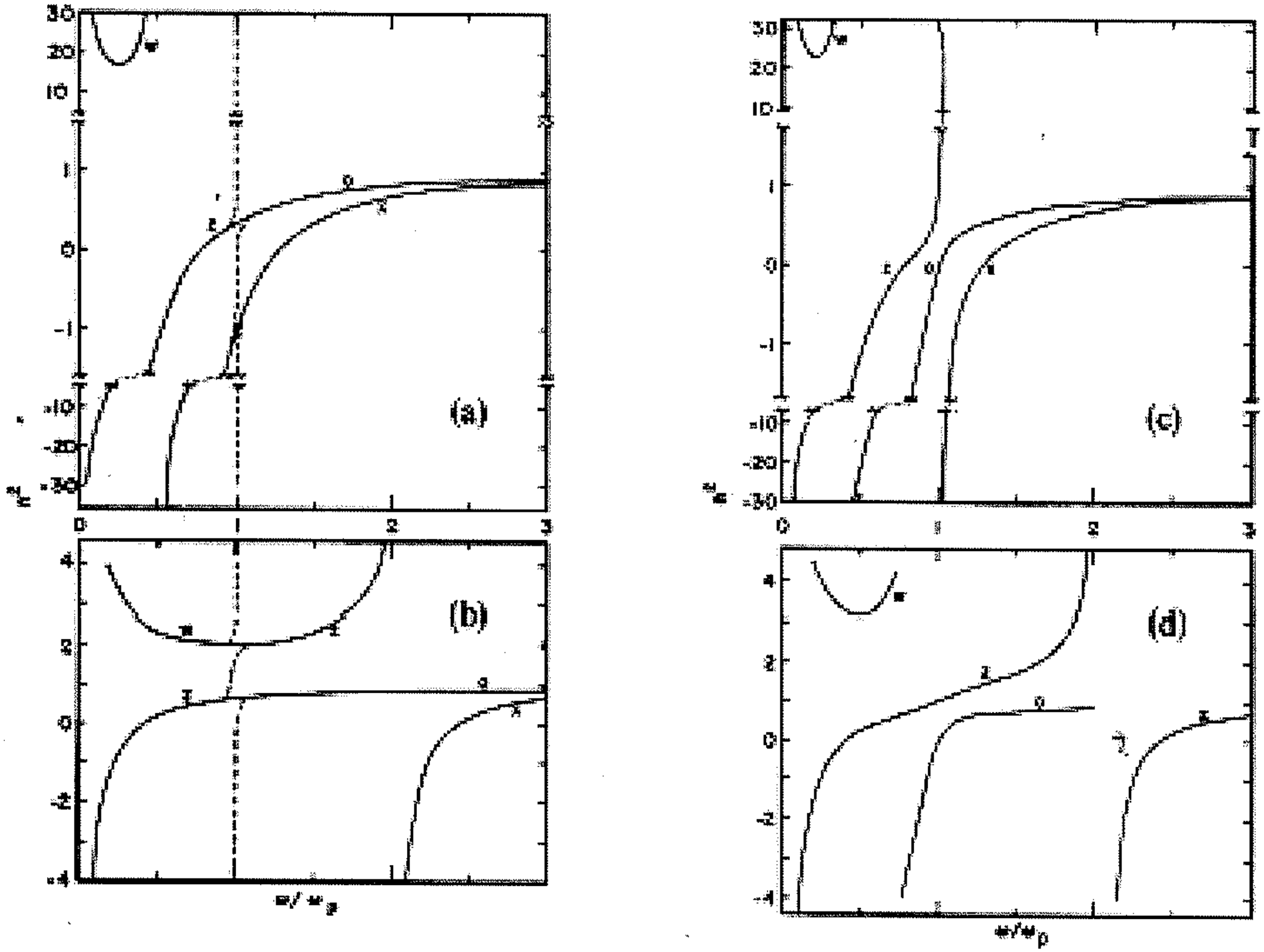
$$\frac{Y\sin^2\theta}{(1-X)\cos\theta}$$

is small or large. If is is large, the propagation regime is referred to as "quasi-longitudinal" or "quasi-circular", whereas if it is small, the propagation regime is referred to as "quasi-transverse" or "quasi-planar". When $X \ll 1$ and $Y \ll 1$, the propagation angle that separates the two regimes is approximately $\theta_o \approx cos^{-1}(Y/2)$. Thus, at high frequencies, the quasi-circular approximation applies for all angles except for a rather narrow range around $\theta = \pi/2$. On the other hand, when $X \approx 1$, the quasi-planar approximation holds except for a small range around $\theta = 0$. Usually, the quasi-circular approximation is valid. When this is the case, the o-mode is left-hand circularly polarized (for $\cos\theta > 0$) and the x-mode is right-hand circularly polarized.

As for Case 1 and Case 2 propagation, the dispersion relation for the general case implies the presence of cutoffs and resonances. The cutoffs are the same as those derived in §2.4; namely, at $\omega = \omega_{pe}$, ω_R, and ω_L. The expression for the resonances is somewhat more complicated in the general case:

$$\omega_\pm^2 = \omega_U H^2 \pm [\omega_{UH}^4 - 4\omega_{pe}^2\omega_{Be}^2\cos^2\theta]^{1/2}$$

The cutoff at $\omega = \omega_{pe}$ and the resonance at $\omega = \omega_-$ apply to the o-mode whereas the cutoffs at $\omega = \omega_{R,L}$ and the resonance at $\omega = \omega_+$ apply to the x-mode. The cutoffs and resonances divide the refractive index curves into four branches, two for each mode. The low frequency branch of the o-mode is called the *whistler* mode and the low frequency branch of the x-mode is called the *z-mode*. The low


(a)
(b)
(c)
(d)
n^2
ω/ω_p

frequency branches are prevented by stop bands from propagating to infinity. The high frequency branches, which retain their name as the x- and o-modes, are the ones of astrophysical interest because they do propagate to infinity. Note that for a medium like the Sun's corona for which the plasma density and magnetic field strength decrease with radius, the parameters X and Y decrease with radius and the x- and o-modes rapidly tend to the high frequency limit.