Basic Plasma Physics Principles

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Outline

Single particle orbits; drifts
Magnetic mirroring
MHD Equations
Force-free fields
Resistive Diffusion
The Vlasov equation; plasma waves

Single particle orbits

E and B fields are *prescribed;* particles are "test particles"

Single particle orbits

$\mathbf{F} = q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B})$

Set $\mathbf{E} = \mathbf{0}$ (for now)

$\mathbf{F} = q\,\mathbf{v} \times \mathbf{B}$

Since $\mathbf{F} \perp \mathbf{v}$, no energy gain ($\mathbf{F}.\mathbf{v} = 0$)

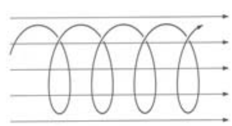
Particles orbit field line

$mv^2/r = qvB$

$r = mv/qB$ (gyroradius)

$\omega = v/r = qB/m$ (gyrofrequency)

Motion in a Uniform Magnetic Field



Is this an electron or ***an ion***?

Drifts

$\mathbf{F} = q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B})$

Now let $\mathbf{E} \neq \mathbf{0}$.

Relativistic transformation of **E** and **B** fields:

$\mathbf{E'} = \gamma(\mathbf{E} + (\mathbf{v}/c) \times \mathbf{B})$

$\mathbf{B'} = \gamma(\mathbf{B} - (\mathbf{v}/c) \times \mathbf{E})$

$E'^2 - B'^2 = E^2 - B^2$

If $E < B$ (so that $E^2 - B^2 < 0$), transform to frame in which $\mathbf{E} = \mathbf{0}$:

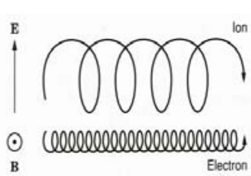
$\mathbf{v} = c\,(\mathbf{E} \times \mathbf{B})/B^2$

In this frame, we get simple gyromotion

So, in 'lab' frame, we get gyro motion, plus a drift, at speed

$\mathbf{v}_D = c\,(\mathbf{E} \times \mathbf{B})/B^2$

E × B drift



Drifts

Exercise:

What if E > B?

Drifts

$\mathbf{v}_D = c\,(\mathbf{E} \times \mathbf{B})/B^2$

E is "equivalent electric field"

Examples:

(1) **E** is actual electric field: $\mathbf{v}_D = c\,(\mathbf{E} \times \mathbf{B})/B^2$ (independent of sign of q)

(2) Pressure gradient: $q\mathbf{E} = -\nabla p$: $\mathbf{v}_D = -c\nabla p \times \mathbf{B}/qB^2$ (dependent on sign of q)

(3) Gravitational field: $m\mathbf{g} = q\mathbf{E}$: $\mathbf{v}_D = (mc/q)\,\mathbf{g} \times \mathbf{B}/B^2$ (dependent on m and sign of q)

Puzzle: in absence of magnetic field, particles subject to **g** accelerate at the same rate and in the same direction; particles subject to **E** accelerate in opposite directions at a rate which depends on their mass. *Why is the exact opposite true when a **B** is present?*

Magnetic Mirroring

Adiabatic invariants

Slow change of ambient parameters: Action $\int p\, dq$ (e.g. Energy/frequency) is conserved

Apply this to gyromotion: $E = (1/2)\, m v_\perp^2$; $\Omega = eB/m$

Then as B slowly changes, $m v_\perp^2/(B/m) = m^2 v_\perp^2/B = p_\perp^2/B$ is conserved

As B increases, $p_\perp$ increases and so, to conserve energy, $p_{//}$ must decrease. This can be expressed as a *mirror force*

$$\mathbf{F} = -\,(p_\perp^2/2m)\,(\nabla B/B),$$

This force causes particles to be trapped in loops with high field strengths at the ends. Note that a magnetic compression also acts as a reflecting wall; this will help us understand particle acceleration later.

Plasma physics in principle

- Solve equations of motion with initial **E** and **B**:
  $$m d^2\mathbf{r}_i/dt^2 = q_i\,(\mathbf{E} + [d\mathbf{r}_i/dt] \times \mathbf{B})$$
- Then use the resulting $\mathbf{r}_i$ and $d\mathbf{r}_i/dt$ to get charge density $\rho(\mathbf{r})$ and current density $\mathbf{j}(\mathbf{r})$
- Then obtain the self-consistent **E** and **B** through Maxwell's equations:
  $$\nabla.\mathbf{E} = \rho$$
  $$\nabla\times\mathbf{B} = (4\pi/c)(\mathbf{j} + \partial\mathbf{E}/\partial t)$$
- "Lather, rinse, repeat"

Plasma physics in principle

- Requires the solution of $\sim 10^{27}$ coupled equations of motion
- ***Not*** a practical method!

MHD Equations

- Replace $\sim 10^{27}$ coupled equations of motion by "averaged" fluid equations
- Neglect displacement current (plasma responds very quickly to charge separation); then body force
  $$\mathbf{F} = (1/c)\,\mathbf{j} \times \mathbf{B} = (1/4\pi)\,(\nabla\times\mathbf{B}) \times \mathbf{B}$$

Complete set of MHD Equations

Continuity: $\partial\rho/\partial t + \nabla.(\rho\mathbf{v}) = 0$

Momentum: $\rho\, d\mathbf{v}/dt = -\nabla p + (1/4\pi)\,(\nabla\times\mathbf{B}) \times \mathbf{B} - \rho\mathbf{g}$

Energy: ?? (can use polytrope: $d(p/\rho^\gamma)/dt = 0$)

Induction: $\partial\mathbf{B}/\partial t = \nabla\times(\mathbf{v} \times \mathbf{B})$

These are 4 equations for the 4 unknowns

(ρ, p, **v**, **B**)

Force – Free Fields

Equation of motion is

$$\rho\, d\mathbf{v}/dt = -\,\nabla p + (1/4\pi)\,(\nabla\times\mathbf{B})\times\mathbf{B}$$

Define the *plasma β* = ratio of terms on RHS = $p/(B^2/8\pi)$

For typical solar corona,

$$p = 2nkT \sim 2(10^{10})(1.38 \times 10^{-16})(10^7) \sim 10$$
$$B \sim 100$$
$$\rightarrow \beta \sim 10^{-3}$$

So second term on RHS dominates, and in steady-state **j** must be very nearly parallel to **B**, i.e.

$$(\nabla\times\mathbf{B})\times\mathbf{B} \cong \mathbf{0}$$

Force – Free Fields

$$(\nabla\times\mathbf{B})\times\mathbf{B} = \mathbf{0}$$

Solutions:

(1) $\mathbf{B} = 0$ (trivial)

(2) $\nabla\times\mathbf{B} = 0$ (current-free – "potential" field)

(3) Linear case: $(\nabla\times\mathbf{B}) = \alpha\mathbf{B}$

(4) Full case: $(\nabla\times\mathbf{B}) = \alpha(\mathbf{r})\mathbf{B}$

Note that taking the divergence of

$$(\nabla\times\mathbf{B}) = \alpha(\mathbf{r})\mathbf{B}$$

gives $0 = \nabla\alpha.\mathbf{B} + \alpha\,\nabla.\mathbf{B}$, so that $\mathbf{B}.\nabla\alpha = 0$, i.e., *α is constant on a field line.*

Resistive Diffusion

Consider the Maxwell equation

$$\nabla\times\mathbf{E} = -\,(1/c)\,\partial\mathbf{B}/\partial t,$$

together with Ohm's law

$$\mathbf{E}_{local} = \mathbf{E} + (\mathbf{v}/c) \times \mathbf{B} = \eta \mathbf{j} = (\eta c/4\pi)\,\nabla\times\mathbf{B}$$

Combined, these give:

$$\partial\mathbf{B}/\partial t = \nabla\times(\mathbf{v} \times\mathbf{B}) - (\eta c^2/4\pi)\,\nabla\times(\nabla\times\mathbf{B}),$$

i.e.

$$\partial\mathbf{B}/\partial t = \nabla\times(\mathbf{v} \times\mathbf{B}) + D\,\nabla^2\mathbf{B},$$

where $D = \eta c^2/4\pi$ is the *resistive diffusion coefficient*.

Resistive Diffusion

$$\partial\mathbf{B}/\partial t = \nabla\times(\mathbf{v} \times\mathbf{B}) + D\,\nabla^2\mathbf{B}$$

The magnetic flux through a given contour S is given by

$$\Phi = \iint_S \mathbf{B}.\, d\mathbf{S}.$$

The change in this flux is given by

$$d\Phi/dt = \iint_S \partial\mathbf{B}/\partial t\ -\int_\Gamma \mathbf{E}.\, d\mathbf{\Gamma},$$

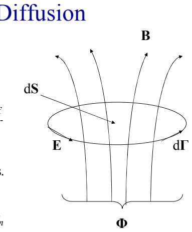
where the second term is due to Faraday's law. If the electric field is generated due to cross-field fluid motions, then, using Stokes' theorem

$$\int_\Gamma \mathbf{E}.\, d\mathbf{\Gamma} = \iint_S \nabla\times\mathbf{E}.\, d\mathbf{S} = \iint_S \nabla\times(\mathbf{v} \times\mathbf{B}).\, d\mathbf{S}$$

we see that

$$d\Phi/dt = \iint_S [\partial\mathbf{B}/\partial t - \nabla\times(\mathbf{v} \times\mathbf{B})]\, d\mathbf{S} = \iint_S D\,\nabla^2\mathbf{B}\, d\mathbf{S}.$$

Thus, if D=0, the field is *frozen in* to the plasma; the flux through an area stays constant as the area deforms due to fluid motions. If, on the other hand, $D \neq 0$, then the flux *can* change (and as a result the energy in the magnetic field can be released).



Resistive Diffusion

$$\partial \mathbf{B}/\partial t = \nabla\times(\mathbf{v}\times\mathbf{B}) + D\,\nabla^2\mathbf{B}$$

The ratio of the two terms on the RHS:

$$|\nabla\times(\mathbf{v}\times\mathbf{B})| / D\,|\nabla^2\mathbf{B}| \sim vL/D \sim 4\pi vL/\eta c^2$$

is known as the *magnetic Reynolds number* S. For $S >> 1$, the plasma is essentially diffusion-free, for $S << 1$ the dynamics are driven by resistive diffusion.

For a flare loop, $V \sim V_A \sim 10^8$ cm s^{-1}, $L \sim 10^9$ cm and $\eta \sim 10^{-7}\,T^{-3/2} \sim 10^{-17}$. This $S \sim 10^{14}$, and the plasma should be almost perfectly frozen in.

The timescale for energy release should be of order $L^2/D \sim 4\pi L^2/\eta c^2$ (this is of order the timescale for resistive decay of current in an inductor of inductance L/c^2 and resistance $R = \eta L/L^2 = \eta/L$). For solar values, this is 10^{15} s $\sim 10^7$ years!

Summary to Date

- Solar loops are big (they have a high inductance)
- Solar loops are good conductors
- Solar loops have a low ratio of gas to magnetic pressure β

So:

The plasma in solar loops is tied to the magnetic field, and the motion of this field determines the motion of the plasma trapped on it

Also:

It is very difficult to release energy from such a high-conductivity, high-inductance system!

The Vlasov Equation

Note that we have still *prescribed* **E** and **B**. A proper solution of the plasma equations requires that **E** and **B** be obtained *self-consistently* from the particle densities and currents. The equation that accounts for this is called the *Vlasov equation*.

Phase-space Distribution Function

This is defined as the number of particles per unit volume of space per unit volume of velocity space:

At time t, number of particles in elementary volume of space, with velocities in range $\mathbf{v} \rightarrow \mathbf{v} + d\mathbf{v} = f(\mathbf{r},\mathbf{v},t)\, d^3\mathbf{r}\, d^3\mathbf{v}$

$f(\mathbf{r},\mathbf{v},t)$ has units cm^{-3} (cm s^{-1})$^{-3}$

The Boltzmann Equation

This equation expresses the fact that the net gain or loss of particles in phase space is due to collisional depletion:

$$Df/Dt \equiv \partial f/\partial t + \mathbf{v}.\nabla f + \mathbf{a}.\nabla_v f = (\partial f/\partial t)_c$$

The Boltzmann equation takes into account the self-consistent evolution of the **E** and **B** fields through the appearance of the acceleration term **a**.

The Vlasov Equation

This is a special case of the Boltzmann equation, with no collisional depletion term:

$$\partial f/\partial t + \mathbf{v}.\nabla f + \mathbf{a}.\nabla_v f = 0,$$

i.e.,

$$\partial f/\partial t + \mathbf{v}.\nabla f + (q/m)\,(\mathbf{E} + (\mathbf{v}/c)\times\mathbf{B}).\nabla_v f = 0.$$

The Electrostatic Vlasov Equation

Setting B = 0, we obtain, in one dimension for simplicity, with q = -e (electrons)

$$\partial f/\partial t + v\,\partial f/\partial x - (eE/m)\,\partial f/\partial v = 0.$$

Perturb this around a uniform density, equilibrium (E = 0) state $f_o = n g_o$:

$$\partial g_1/\partial t + v\,\partial g_1/\partial x - (eE_1/m)\,\partial g_o/\partial v = 0.$$

Also consider Poisson's equation ($\nabla.\mathbf{E} = 4\pi\rho$):

$$\partial E_1/\partial x = 4\pi\rho = -4\pi ne \int g_1\, dv$$

The Electrostatic Vlasov Equation

Now consider modes of the form

$$g \sim \exp(i[kx-\omega t])$$

Then the Vlasov equation becomes

$$-i\omega g_1 + ivkg_1 - (eE_1/m)\, dg_o/dv = 0$$

$$(\omega - kv)g_1 = (ieE_1/m)\, dg_o/dv$$

and Poisson's equation is

$$ikE_1 = -4\pi ne \int g_1\, dv$$

Combining,

$$ikE_1 = -i(4\pi ne^2/m)\, E_1 \int dg_o/dv\, dv/(\omega - kv)$$

Simplifying, and defining the *plasma frequency* through $\omega_{pe}^2 = 4\pi ne^2/m$,

$$1 - (\omega_{pe}^2/k^2) \int dg_o/dv\, dv/(v - \omega/k) = 0.$$

This is the *dispersion relation* for electrostatic plasma waves.

The Electrostatic Vlasov Equation

Integrating by parts, we obtain an alternative form

$$1 - (\omega_{pe}^2/\omega^2) \int g_o \, dv/(1 - kv/\omega)^2 = 0.$$

For a *cold* plasma, $g_o = \delta(v)$, so that we obtain

$$1 - (\omega_{pe}^2/\omega^2) = 0, \text{ i.e., } \omega = \omega_{pe}$$

The Electrostatic Vlasov Equation

For a warm plasma, we expand the denominator to get

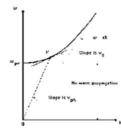
$$1 - (\omega_{pe}^2/\omega^2) \int g_o \, dv[1 + 2kv/\omega + 3k^2v^2/\omega^2 + \ldots] = 0$$

i.e. $1 - (\omega_{pe}^2/\omega^2)\,[1 + 3k^2\langle v\rangle^2/\omega^2 + \ldots] = 0$,

where $\langle v\rangle^2 = k_BT/m$ is the average *thermal speed*. This gives the *dispersion relation*

$$\omega^2 = \omega_{pe}^2 + 3\,(k_BT/m)\,k^2$$

(cf. $\omega^2 = \omega_{pe}^2 + c^2k^2$ for EM waves)



Dispersion relations

Electrostatic waves in a warm plasma:

$$\omega^2 = \omega_{pe}^2 + 3\,(k_BT_e/m)\,k^2$$

Ion-acoustic waves (includes motion of ions):

$$\omega = kc_s;\ c_s = [k_B(T_e + T_i)/m_i]^{1/2}$$

(note electrons effectively provide quasi-neutrality)

Upper hybrid waves (includes B):

$$\omega^2 = \omega_{pe}^2 + \Omega_e^2;\ \Omega_e = eB/m_e$$

Dispersion relations

Alfvén waves:

$$\omega^2 = k^2V_A^2/[1 + (V_A^2/c^2)]$$

Magnetoacoustic waves:

$$\omega^4 - \omega^2k^2(c_s^2 + V_A^2) + c_s^2V_A^2k^4\cos^2\theta = 0$$

(θ = angle of propagation to magnetic field)

etc., etc.

Two-Stream Instability

$$1 - (\omega_{pe}^2/\omega^2) \int g_o \, dv/(1 - kv/\omega)^2 = 0.$$

For two streams,

$$g_o = [\delta(v-U) + \delta(v+U)],$$

so that

$$(\omega_{pe}^2/[\omega-kU]^2) + (\omega_{pe}^2/[\omega+kU]^2) = 1.$$

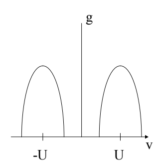
This is a quadratic in ω^2:

$$\omega^4 - 2(\omega_{pe}^2 + k^2U^2)\omega^2 - 2\,\omega_{pe}^2k^2U^2 + k^4U^4 = 0,$$

with solution

$$\omega^2 = (\omega_{pe}^2 + k^2U^2) \pm \omega_{pe}\,(\omega_{pe}^2 + 4k^2U^2)^{1/2}$$

There are solutions with ω^2 negative and so imaginary (exponentially growing) solutions.



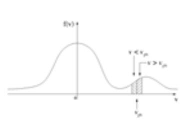
Two-Stream Instability

Distribution with two maxima (one at zero, one at the velocity of the "beam") is susceptible to the two-stream instability.

This generates a large amplitude of plasma waves and affects the energetics of the particles.

Two-Stream Instability

This can also happen due to an "overtaking" instability – fast particles arrive at a location earlier than slower ones and so create a local maximum in f.



Summary

High energy solar physics is concerned with the physics of plasma, which is a highly interacting system of particles and waves.

"Plasma physics is complicated" (J.C. Brown & D.F. Smith, 1980)