STEADY-STATE FOKKER-PLANCK CODE

DOCUMENTATION

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Last Revised August 10, 2001

Introduction & Summary

This program computes the steady-state electron distribution function in a flare magnetic loop as a function of position along a magnetic field line, electron pitch angle, and electron energy. Subroutine ELECTRONFLUX computes the distribution function everywhere along the field line, given the injected flux at the top of the field line and a model solar atmosphere. The steady-state Fokker-Planck equation is integrated. Coulomb and synchrotron energy losses and pitch angle diffusion are included, as well as magnetic mirroring.

To run the program, the following files must be compiled and linked:

efluxprog.f
electronflux.f
efluxsubs.f
ssfpinc.f

Input parameters are specified in Program EFLUXPROG and in the text file "efluxin.txt". The output is written to the file "efluxout.txt". The IDL program "edistplot.pro" reads the output from "efluxout.txt" and plots the results.

The file "efluxprog.f" contains the main Program EFLUXPROG and subroutines that set up the numerical mesh and the flare model.

"Electronflux.f" contains the Fokker-Planck equation integration routine, Subroutine ELECTRONFLUX.

"Efluxsubs.f" contains subroutines called by Subroutine ELECTRONFLUX.

"Ssfpinc.f" is an include file used by the routines in "efluxprog.f". It specifies the values of common parameters, types variables, dimensions arrays, and sets up common blocks.

Contacts & Acknowledgements

This program was originally developed by John Leach and Vahé Petrosian (ApJ 251, 781, 1981), based upon a program developed by Martin Walt, William MacDonald, and William Francis (described in *Physics of the Magnetosphere*, ed. R. Carovillano and J. F. McClay, New York: Reinhold, p. 534, 1968). The code was extended to ultrarelativistic energies by Jim McTiernan and Vahé Petrosian (ApJ 359, 524, 1990). The routines in "efluxprog.f" have been modified by Gordon Holman for this public distribution of the code.

Address questions and comments about the routines in the file "efluxprog.f" to

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Subroutines in efluxprog.f

The subroutines described below are used by Program EFLUXPROG to set up the numerical mesh and physical model for the Fokker-Planck equation integrator, Subroutine ELECTRONFLUX in the file "electronflux.f".

atmosphere	Calculates the atmospheric densities of the constituents as a function of height and position along the magnetic field line (y). Also computes the total density and column depth as a function of y.
coordmap	Relates position along the magnetic field line (y) to height in the solar atmosphere.
edistin	Provides the injected electron distribution at y = 0, fz(i), as a function of energy, z(i) (electrons/cc/keV). (Called by Subroutine fluxin)
fluxin	Provides the injected electron flux at y = 0, ftop(i,k), as a function of electron energy and pitch angle.
fpinput	Reads the input file "efluxin.txt".
fpoutput	Writes the output file "efluxout.txt".
magfield	Provides the magnetic field strength as a function of position along the magnetic field line (y).
mesh	Provides the coordinate mesh for pitch angle, x(k), position along the magnetic field line, y(j), and electron energy, z(i).
padistin	Provides the injected electron distribution function at y = 0, xfm(k), as a function of pitch angle, x(k). (Called by Subroutine fluxin)

Subroutine Electronflux Users' Guide

Subroutine Electronflux returns the steady-state Fokker-Planck solution for a beam of electrons in a solar loop. The solution includes effects due to Coulomb collisions with the ambient plasma, effects due to synchrotron emission by the electrons, and effects of bremsstrahlung emission. (The synchrotron and bremsstrahlung effects are only noticeable at high energies, above about 10 MeV.) The routine is designed to be used for electron energies from a few keV up to hundreds of MeV. It has been tested over this energy range, for ambient densities from 1.0e8 to 1.0e24, and for magnetic field strengths up to 10000 Gauss.

Details can be found in McTiernan and Petrosian, ApJ 359, p. 524, (1990), and also in Leach and Petrosian, ApJ 251, p. 781, (1981). Excerpts from McTiernan (1989, Ph. D. thesis, Stanford U., Chapter 2), which describe the equation, are included in Appendix A. The method of numerical solution is described in Appendix B (McTiernan 1989, Ph. D. thesis, Stanford U.).

There are two files included, "electronflux.f" and "efluxsubs.f". "Electronflux.f" does the solution, using various subroutines defined in "efluxsubs.f". The user is expected to provide all of the inputs to electronflux. Once a main program is defined, just compile and link:

```
F77 main.f other_user_routines.f electronflux.f efluxsubs.f
```

(Note that these routines are written in standard Fortran 77. There should be no difficulty compiling in Fortran 90.)

The solution is solved on a 3-D grid of energy, distance from the loop top, and pitch angle (pitch angle is the angle of the direction of motion of an electron relative to the magnetic field).

The user is expected to provide the model of the solar atmosphere and the electron flux at the top of the loop, along with all of the information about the grid for the solution. The function call is:

CALL ELECTRONFLUX (KMAX, X, JMAX, Y, IMAX, Z, NC, ZA, ZN, EI, DN, BF, FTOP, FLUX, KB, KB1, KHS, KBC, KSY, KSMU, KTRM, KITM, KKR, X1, ADP, ADPM)

An explanation of each input parameter follows.

First, the numerical grid parameters:

KMAX = number of pitch angle grid points, up to a maximum of 30.

Even numbers are preferred; the code has never been used with an odd number of pitch angle grid points.

X = COS (PITCH ANGLE), the pitch angle grid, from 1 to –1. No zeros are allowed; in the steady-state solution, 0 pitch angle is a singular point.

JMAX = number of depth steps, up to 200

Y = distance to bin edges in cm (Y must start at 0.0), measured from the top of the loop. There is a maximum of 200 distance steps allowed, so the maximum number of elements allowed in the Y array is 201.

IMAX = number of electron energies, up to 120

Z = electron energy in units of mc^2, up to 120 elements

(Note that the maximum sizes for these arrays can be changed via parameter statements in the code:

parameter (ndy=200,ndy1=201,ndx=30,ndz=120,ndc=12,ndxy=6000)

NDX=30 is the maximum number of pitch angle points, NDY=200 is the maximum number of distance steps, and NDZ=120 is the maximum number of electron energies.)

Next, the atmospheric model:

NC = number of atmospheric constituents in the model. There can be as many as 12. Constituent 3 is electrons, constituents 4 to NC are ions. Typically, 1 is Hydrogen, 2 is Helium, 3 is electrons (demanded), 4 is protons. Most simulations in the past have been run with pure hydrogen, so only constituents 1, 3 and 4 are used in those cases. But this is not a requirement.

ZA = the atomic number for neutral atmospheric constituents. For ions, ZA is the charge number (i.e. 1 for an ion with 1 missing electron, 2 for an ion with 2 missing electrons, etc.).

ZN = molecular weight

EI = the ionization energy, in keV, e.g. 11.5 for H, 41.5 He, 0.0 for electrons and ions.

DN = the density of each constituent at points Y, max. size of (12, 201).

BF = the magnetic field in gauss for each Y.

Next, the input electron distribution:

FTOP = the electron flux distribution at the top of the loop, for all energies and pitch Angles. The x=1 element must be nonzero for all energies. The units of FTOP are electrons/(cm^2-ster-sec-mc^2).

Next, control parameters for the solution (generally, if you set one of these to 0, the appropriate term is not included in the solution):

KB – if 0 then no Coulomb energy loss terms are included, if 1 then Coulomb energy losses are included

KB1 – if 0 no Coulomb pitch angle diffusion, if 1 keep diffusion

KHS – if 0 no magnetic mirroring, if 1 keep mirroring

KBC – if 0 no reflection at top, if 1, reflect

KSY – if 0 no synchrotron energy losses or diffusion, if 1 include synchrotron energy losses

KSMU – if 0 no synchrotron pitch angle diffusion, if 1 include synchrotron pitch angle diffusion

KTRM – if 0, undesired negative values are removed while solving for FLUX

Next, iteration control parameters with some default values:

KITM = the number of iterations allowed per energy, max = 9999. A good default value is 20.

KKR = the number of iterations between divergence checks, from 1 to 9999. A good default value is 3.

X1 = the value of tau/eta above which the solution is set to zero. Tau/eta is a normalized column depth, see McTiernan and Petrosian, ApJ 359, p524. For large values of tau/eta the solution can become unstable. A good default value is 30.

ADP = degree of relative accuracy desired (3E-7 = reasonable default).

ADPM = degree of relative accuracy required (5E-1 = reasonable default).

Finally, the output:

FLUX = electron flux distribution at all points in the loop, for all angles and energies. The units are electrons/(cm^2-ster-sec-keV).

The Input File

efluxin.txt with labels

nc nh

4 2

zn	za	ei	
1.00	1.00	11.50	H
4.00	2.00	41.50	He
0.00	1.00	0.00	e^-
1.00	1.00	0.00	p+

h	dni (H)	dni (He)	dni (e^-)	dni (p^+)
1.0000E+06	0.0	0.0	2.0000E+11	2.0000E+11
-2.9000E+02	0.0	0.0	6.2586E+21	6.2586E+21

kx jy iz

24 7 49

x (cosine theta array)

1.000	0.995	0.970	0.927	0.866	0.788
0.695	0.588	0.469	0.342	0.208	0.070
-0.070	-0.208	-0.342	-0.469	-0.588	-0.695
-0.788	-0.866	-0.927	-0.970	-0.995	-1.000

ydy

0.000 129.300 7758.000 430.200 33570.000 4.754 33860.000

zdz

0.016	0.004	0.028	0.006	0.040	0.008
0.056	0.012	0.080	0.016	0.112	0.022
0.156	0.027	0.210	0.040	0.250	0.050
0.300	0.075	0.600	0.100	1.000	0.160
1.640	0.240	2.600	0.360	4.040	0.640
6.600	1.000	11.600	1.600	16.400	2.400
26.000	3.600	40.400	6.400	66.000	10.000
116.000	16.000	164.000	24.000	260.000	36.000
404.000					

CHAPTER 2

THE EQUATION AND ANALYTIC SOLUTIONS

In this chapter, we discuss the kinetic equation and find analytic solutions in some simple limiting cases. While the results presented here are general and can be used for problems other than the solar flare problem the justifications for the approximations are given for solar flare conditions. The application of these results to other situations (for example, to neutron star γ-ray bursts) must be accompanied by a justification for the approximations under the proper conditions.

We use the Fokker-Planck method for the solution as described by Leach and Petrosian (1981, hereafter known as LP). The full equation is given in Section 2.1 and justifications for the approximations used are given in Section 2.2. We solve the steady-state equation, including the effects of collisions and synchrotron emission. The effects of a converging magnetic field are also included. The reverse current is neglected; this is a good approximation for small fluxes of non-thermal electrons. Also we neglect the effects of synchrotron absorption and inverse Compton emission. In Section 2.3 we describe the analytic solutions in the cases for weak uniform magnetic field, zero density and uniform magnetic field, negligible diffusion and converging magnetic field with negligible synchrotron and collisional losses. These solutions are complex and are given in full in Appendix A.

2.1 THE EQUATION

We present here the fully relativistic kinetic equation for non-thermal electrons including synchrotron losses and the effects of collisions (taken into account by the Fokker-Planck method). The equation can be written

$$\frac{df}{dt} = \left.\frac{df}{dt}\right|_{\rm coll} - \frac{\partial}{\partial E}(\dot{E}f) - \frac{\partial}{\partial \mu}(\dot{\mu}f) + \Sigma \ , \qquad (2.1.1)$$

where $\dot{E}$ and $\dot{\mu}$ denote the rates of change in the energy and pitch angle cosine due to processes which are not included in the usual Fokker-Planck equation.

The collisional terms were given by LP (cf. also Leach 1984), and we may write

$$\frac{\partial f}{\partial t} + \mu\beta c\frac{\partial f}{\partial s} = c\frac{\partial}{\partial E}\left(\frac{Cf}{\beta}\right) + \frac{\xi Cc}{\beta^3\gamma^2}\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial f}{\partial \mu}\right] - \frac{\partial}{\partial E}(\dot{E}_s f) - \frac{\partial}{\partial \mu}(\dot{\mu}_s f) - \frac{\partial}{\partial \mu}(\dot{\mu}_{cv} f) + \Sigma \ , \tag{2.1.2}$$

where

s = depth along field lines,

E = electron kinetic energy in units of $m_e c^2$,

$\mu = \cos(\alpha)$, α = pitch angle,

$\gamma = E + 1$ is the total energy, $\beta = v/c$ where v is the velocity, B is the magnetic field strength and $\Sigma(E, \mu, s)$ is a source term for the injected electrons.

The function $f(E, \mu, s)$ is the electron distribution function so that the electron beam density at the depth s is

$$\int dE \int d\mu f(E, \mu, s) = n_b(s) \ , \tag{2.1.3}$$

We have assumed azimuthal symmetry about the field lines, so that we only need the one spatial coordinate s.

The collisional energy loss rate is given by $\dot{E} = -Cc/\beta$, but denoting $\dot{\mu}_c$ is complex since the equation includes diffusion in pitch angle and the diffusion term is not of the simple form $\partial(\dot{\mu}_c f)/\partial\mu$. We will use $\dot{\mu} = \xi Cc(1-\mu^2)/\beta^3\gamma^2$ in our comparisons. The quantities C and ξC are given by (Evans 1955, Snyder and Scott 1949)

$$C = 4\pi r_o^2 (n_e \ln\Lambda + \sum_n Z_n n_n \ln\Lambda'_n)$$

and

$$\xi C = 2\pi r_o^2 \left(n_e \ln\Lambda + \sum_i Z_i^2 n_i \ln\Lambda + \sum_n Z_n^2 n_n \ln\Lambda_n'' \right) \quad , \tag{2.1.4}$$

where Z_i and n_i denote the charge number ($Z_i = q/e$) and density of the ith ionized atmospheric component and Z_n and n_n are the atomic number and density of the nth neutral component. The quantity $r_o = e^2/m_e c^2$ is the classical electron radius, m_e is the electron mass and e is the electron charge. The $\ln\Lambda$'s are Coulomb collision logarithms, which for a cold target are given by

$$\Lambda^2 = \left(\frac{\beta^4 \gamma}{\pi \alpha n_e} \left(\frac{m_e c}{\hbar} \right)^3 \right) \quad , \tag{2.1.5}$$

$$\Lambda_i'^2 = \beta^2 \gamma^2 (\gamma - 1)/I_i^2 \quad , \tag{2.1.6}$$

and

$$\Lambda_i''^2 = \beta^2 \gamma^2 / 2 Z_i^{2/3} \alpha^2 \quad . \tag{2.1.7}$$

Here α is the fine structure constant and I_i is the ionization energy of the ith component in units of $m_e c^2$, and Z_i is its atomic number. Some numerical values for the Coulomb logarithms for different electron energies, and for ambient protons, electrons and neutral hydrogen and helium are given in Table 2.1.

Table 2.1

Values of the Coulomb logarithms used in the definitions of C and $C' = \xi C$ for different energies. The values of $\ln\Lambda$ are given for $n_e = 10^{10}\mathrm{cm}^{-3}$.

E(keV)	$\ln\Lambda$	$\ln\Lambda_{\mathrm{H}}'$	$\ln\Lambda_{\mathrm{He}}'$	$\ln\Lambda_{\mathrm{H}}''$	$\ln\Lambda_{\mathrm{He}}''$
16	22.96	7.43	6.31	3.20	2.96
300	25.49	10.48	9.36	4.78	4.55
10,600	27.30	15.13	14.01	7.65	7.42

The rate of change of the electron pitch angle due to the change in the magnetic field, $\dot{\mu}_{cv}$, is derived from consideration of the adiabatic invariance of the magnetic moment of an electron in a slowly varying magnetic field, i.e., (Jackson 1962, Chapter 12)

$$\frac{\beta^2\gamma^2(1-\mu^2)}{B} = \text{constant} \ . \tag{2.1.8}$$

The magnetic field does no work on the electron, so $\dot{E}_{cv} = 0$ and

$$\begin{aligned} \dot{\mu}_{cv} &= \frac{d\mu}{dB}\frac{dB}{ds}\frac{ds}{dt} \\ &= -\beta c\frac{(1-\mu^2)}{2}\frac{d\ln B}{ds} \ . \end{aligned} \tag{2.1.9}$$

The rates of change $\dot{E}_s$ and $\dot{\mu}_s$ due to synchrotron emission may be derived by consideration of the radiation reaction force on a particle which emits synchrotron radiation (cf. Petrosian 1985) and are given by

$$\dot{E}_s = -cS\beta^2\gamma^2(1-\mu^2)$$

and

$$\dot{\mu}_s = cS\mu(1-\mu^2)/\gamma \quad , \tag{2.1.10}$$

where $S = 2r_o^2B^2/3m_ec$ is an inverse scale length for synchrotron losses and m_e is the electron mass.

To obtain the equation we eventually solve we define $\Phi = f/\beta$ and divide Equation (2.1.2) by $c\beta^2$ and the fully relativistic equation including collisional and synchrotron effects as well as the effects of the field inhomogeneities can then be written

$$\begin{aligned} \frac{1}{c\beta}\frac{\partial\Phi}{\partial t} + \mu\frac{\partial\Phi}{\partial s} - \frac{d\ln B}{2ds}\frac{\partial}{\partial\mu}[(1-\mu^2)\Phi] &= \frac{1}{\beta^2}\frac{\partial}{\partial E}\left[\left(C + S\beta^3\gamma^2(1-\mu^2)\right)\Phi\right] \\ -\frac{S}{\beta\gamma}\frac{\partial}{\partial\mu}[\mu(1-\mu^2)\Phi] + \frac{\xi C}{\beta^4\gamma^2}\frac{\partial}{\partial\mu}&\left[(1-\mu^2)\frac{\partial\Phi}{\partial\mu}\right] + \frac{\Sigma}{c\beta^2} \ . \end{aligned} \tag{2.1.11}$$

This is the equation we will solve.

2.2 THE APPROXIMATIONS

We have simplified the equation by using the steady-state approximation. The speed of a typical electron is $v \gtrsim 0.3c$, and the size of a flaring loop is typically $L \gtrsim 10^9$cm. It takes a time of $t = L/v \lesssim 1$ second for the lowest energy electrons to travel down the loop. For an ambient density of $n = 10^{10}\text{cm}^{-3}$, the collision time for a typical electron of energy $E \approx 20$ keV is also less than a second, which is much smaller than the duration bursts (> 10 s). Thus we need to have a continuous injection of electrons to sustain the burst. We assume that the injected flux is constant in time so that the solution will be time independent, and we drop the $\partial\Phi/\partial t$ term. This is valid for all time scales greater than the transport time scales mentioned above.

Our calculations include the effects of a B field which changes along the field lines. We neglect any effects due to the curvature of the field lines since the gyroradius is much smaller than the radius of curvature of the loop. The gyroradius of a 16 keV electron is $a = 42(B/100\text{ G})^{-1}$cm and that for a 10.6 MeV electron is $a = 3700(B/100\text{ G})^{-1}$cm. Both of these values are much smaller than the radius of curvature, which is of the order of 10^9cm. So the electrons see a straight magnetic field. The velocity drift due to curvature of the field lines is given by (see Jackson 1962, Ch. 12) $v_{CD} \approx (\mu^2 v)(a/r)$, and r is the radius of curvature of the field lines. We have shown that $a \ll r$, so it is clear that $v_{CD} \ll v$.

If the field has a transverse gradient, which must be true if the field has a parallel gradient, there is a drift across the field lines with a velocity given by $v_{\nabla \text{B}} \approx v\sqrt{(1-\mu^2)}(a/L_{\nabla \text{B}})$ where $L_{\nabla \text{B}} = (|\nabla \mathbf{B}|/B)^{-1}$ is the length scale of the change in B which will be at least of the order of 10^8 cm. Since $a \ll L_{\nabla \text{B}}$, $v_{\nabla \text{B}} \ll v$ and we can ignore the transverse drift also. Thus the electrons are effectively tied to the field lines. Also, since $L_{\nabla \text{B}} \gg a$, we are justified in the use of the adiabatic invariance of the magnetic moment to find $\dot{\mu}_{cv}$.

In order to use the approximations for $\dot{E}_s$ and $\dot{\mu}_s$, we must have the energy loss rate due to synchrotron radiation small in comparison to the electron gyrofrequency, or $\dot{E}_s/E \ll \nu_b$, where $\nu_b = eB/2\pi m_e c$, which leads to the condition

$$B \ll 10^{14}\mathrm{G}/\beta^2\gamma \ , \tag{2.2.1}$$

which is true in most astrophysical situations, specifically for solar conditions where $B \lesssim 1000$ G and $\beta^2\gamma \lesssim 100$ at the most.

We neglect the effects of the reverse current which is set up in the ambient plasma to preserve charge neutrality. We can do this for small values of the beam density n_b or downward beam flux $F_b(s) = \int \mu v f d\mu dE$. If the drift velocity of the reverse current is comparable to the thermal velocity of the ambient electrons, instabilities are generated which rapidly decelerate the plasma. Hence $v_{rev} \ll v_{th}$ must be true. Invoking charge neutrality, we find that $v_{rev} = (F_b/n_e)$. If v_b is the mean velocity of the beam electrons, then $F_b = n_b v_b$; typically $v_b \approx c/3$ and $v_{th} \approx c/20$ (for a 10^7K plasma), so for $v_{rev} \ll v_{th}$ we need $n_b/n_e \ll 0.15$.

The reverse current sets up an electric field which acts to decelerate downward moving beam electrons; $\mathcal{E} = \eta j_{rev}$ where η is the resistivity. Emslie (1980) gave the energy loss and pitch angle terms resulting from the reverse current, and they are as follows:

$$\dot{E}_{rev} = -\frac{\beta\mu e\mathcal{E}}{m_e c} = -\frac{\beta\mu\eta e^2 F_b}{m_e c}$$

and

$$\dot{\mu}_{rev} = -\frac{(1-\mu^2)e\mathcal{E}}{\beta\gamma m_e c} = -\frac{(1-\mu^2)\eta e^2 F_b}{\beta\gamma m_e c} \ , \tag{2.2.2}$$

where η is the resistivity (in e.s.u.), given by

$$\eta = \frac{7.26\times 10^{-9}x}{T^{3/2}} \ln\left(\frac{3}{2e^3}\left(\frac{k^3T^3}{\pi n_e}\right)^{1/2}\right) + 7.6\times 10^{-18}T^{1/2}\frac{(1-x)}{x} \ . \tag{2.2.3}$$

The level of ionization is $x = n_e/n_T$; we will let $x = 1$ since we expect the largest effects at the top of the loop where F_b is largest.

We can calculate $R_{rev} \equiv (\dot{E}_{rev}/\dot{E}_c)$, the ratio of reverse current losses to collisional losses, and we find

$$R_{rev} = \frac{\eta \mu \beta F_b}{4\pi r_o n_e \ln \Lambda} \ . \tag{2.2.4}$$

Adding typical values we find

$$R_{rev} = 1.02 \times 10^{-18} \frac{\beta^2 \mu F_b}{(n_e/10^{10}\mathrm{cm}^{-3})} \ , \tag{2.2.5}$$

so, in order to be able to ignore the deceleration due to the electric field, we need to have $F_b \ll 10^{18}(n_e/10^{10}\mathrm{cm}^{-3})$ or, noting that $F_b = n_b v_b$, $n_b/n_e \ll 10^{-2}$. This is a stricter requirement than the previous one, and there exists a small regime $0.01 \ll n_b/n_e \ll 0.15$ for which the reverse current can have an effect on the beam without causing instabilities in the plasma.

Brown and Melrose (1977) gave typical values for the total number of radiating electrons N for X-ray bursts. They found $N \approx 10^{34}$ which gives a beam density of $n_b \approx 10^6 \mathrm{cm}^{-3}$, assuming that the total flare volume is the length times the area of the emission region ($A \approx 10^{18}\mathrm{cm}^2$). So for typical flares we can neglect the reverse current. However, large flares can have N as large as 10^{39} (Emslie 1980), and in that case we can have $n_b \approx 10^{10}\mathrm{cm}^{-3}$, and unless the ambient density is very large ($n_e \gtrsim 10^{12}\mathrm{cm}^{-3}$) or the area is large ($A \gtrsim 10^{20}\mathrm{cm}^2$), we would expect the reverse current to be important.

We neglect the effects of inverse Compton emission also. The rates of change for inverse Compton emission, $\dot{E}_{IC}$ and $\dot{\mu}_{IC}$ are the same as those for synchrotron emission with B^2 replaced by $8\pi\mathcal{E}_\gamma$, where $\mathcal{E}_\gamma$ is the soft photon energy density at the surface of the sun, which is approximately $L_\odot/4\pi R_\odot^2 c$. Thus the ratio of inverse Compton emission to synchrotron emission is

$$\frac{\dot{E}_{IC}}{\dot{E}_s} = \frac{8\pi\mathcal{E}_\gamma}{B^2} \approx \left(\frac{7.4\ \mathrm{G}}{B}\right)^2 \ , \tag{2.2.6}$$

which is much less than one for $B \gtrsim 100$ G.

2.3 ANALYTIC SOLUTIONS

The fully relativistic equation including collisional and synchrotron effects as well as the effects of the field inhomogeneities can be written

$$\mu\frac{\partial\Phi}{\partial s} - \frac{d\ln B}{2ds}\frac{\partial}{\partial\mu}[(1-\mu^2)\Phi] = \frac{1}{\beta^2}\frac{\partial}{\partial E}\left[\left(C + S\beta^3\gamma^2(1-\mu^2)\right)\Phi\right] - \frac{S}{\beta\gamma}\frac{\partial}{\partial\mu}[\mu(1-\mu^2)\Phi] + \frac{\xi C}{\beta^4\gamma^2}\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\Phi}{\partial\mu}\right] + \frac{\Sigma}{c\beta^2}\ , \qquad (2.3.1)$$

where $\Phi \equiv f/\beta$, $\gamma = E + 1$ is the total energy, $\beta c = c\sqrt{1-1/\gamma^2}$ is the electron velocity, B is the magnetic field strength and Σ is a source term for the injected electrons.

The the steps leading to Equation (2.3.1) and the definition of the collisional energy loss and diffusion coefficients C and $C' = \xi C$, and the synchrotron coefficient S have been given in Section 2.1 and some numerical values for the collisional and synchrotron energy loss and pitch angle change terms are given in Table 2.2. For a background plasma of fully ionized hydrogen, $\xi = 1$ and

$$C = 4\pi r_o^2 n \ln\Lambda = 2\times 10^{-13}\left(\frac{\ln\Lambda}{20}\right)\left(\frac{n}{10^{10}\mathrm{cm}^{-3}}\right)\ \mathrm{cm}^{-1}\ , \qquad (2.3.2)$$

where $r_o = e^2/m_e c^2$ is the classical electron radius, n is the ambient proton or electron density, and $\ln\Lambda$ is the coulomb logarithm. This simple relation is not true for a partially ionized plasma or a neutral gas. In these cases the ratio ξ depends on the energy; e.g., for a neutral background ξ varies from $\sim 1/12$ to $\sim 1/8$ for energies from 10 keV to 10 MeV.

The synchrotron energy loss and pitch angle change terms are proportional to

$$S = \frac{2r_o^2}{3}\left(\frac{B^2}{m_e c^2}\right) = 6.5\times 10^{-16}\left(\frac{B}{100\ \mathrm{G}}\right)^2\ \mathrm{cm}^{-1}\ . \qquad (2.3.3)$$

Table 2.2

Some numerical values for the energy loss and pitch angle change rates.

	COLLISIONS	
	relativistic ($\gamma \gg 1$)	non-relativistic ($\beta^2 \ll 1$)
$(-\dot{\gamma}/\gamma)_c$	$3 \times 10^{-4} \left(\frac{\ln \Lambda}{20}\right) \left(\frac{n}{10^{10}\mathrm{cm}^{-3}}\right) \left(\frac{20}{\gamma}\right)$	$\frac{6 \times 10^{-3}}{\beta} \left(\frac{\ln \Lambda}{20}\right) \left(\frac{n}{10^{10}\mathrm{cm}^{-3}}\right)$
$(\dot{\mu}/\mu)_c$	$1.5 \times 10^{-5} \left(\frac{\ln \Lambda}{20}\right) \left(\frac{n}{10^{10}\mathrm{cm}^{-3}}\right) \left(\frac{20}{\gamma}\right)^2$	$\frac{6 \times 10^{-3}}{\beta^3} \left(\frac{\ln \Lambda}{20}\right) \left(\frac{n}{10^{10}\mathrm{cm}^{-3}}\right)$
	SYNCHROTRON	
	relativistic ($\gamma \gg 1$)	non-relativistic ($\beta^2 \ll 1$)
$(-\dot{\gamma}/\gamma)_s$	$4 \times 10^{-4}(1-\mu^2) \left(\frac{\gamma}{20}\right) \left(\frac{B}{100\ \mathrm{G}}\right)^2$	$2 \times 10^{-5}\beta^2(1-\mu^2) \left(\frac{B}{100\ \mathrm{G}}\right)^2$
$(\dot{\mu}/\mu)_s$	$1 \times 10^{-6}(1-\mu^2) \left(\frac{20}{\gamma}\right) \left(\frac{B}{100\ \mathrm{G}}\right)^2$	$2 \times 10^{-5}(1-\mu^2) \left(\frac{B}{100\ \mathrm{G}}\right)^2$

The coefficients S, C, and C' have the units of inverse length and are useful scales. We will find it convenient to define the ratios $R_o \equiv S/C$ ($R_o = 3230\, B^2/n$ for $\ln \Lambda = 20$), and

$$R = \beta^3 \gamma^2 (1-\mu^2) R_o \ . \tag{2.3.4}$$

to measure of relative importance of the synchrotron and collisional energy losses (see Table 2.2).

For the source term Σ we shall assume that the electrons are injected at one point, the origin of the spatial coordinate $s = 0$, so that $\Sigma \propto \delta(s)$ is zero everywhere except at $s = 0$. Consequently we solve the equation with $\Sigma = 0$ and use the injected distribution as a boundary condition. Furthermore, we will present the results in terms of the particle flux $F = \beta c f = \beta^2 c \Phi$ instead of f or

Φ. The boundary condition at $s = 0$ we assume to have the form

$$F(E, \mu, s = 0) = F_o(E)G(\mu) = F_o(E)\, 2\alpha_o^{-2} e^{-\alpha^2/\alpha_o^2} \ . \tag{2.3.5}$$

Here α_o^2 is the dispersion in pitch angle and is smaller for distributions beamed along the field lines. In some cases we shall replace α^2 with $\sin^2 \alpha = (1 - \mu^2)$ in Equation (2.3.5), and when necessary we shall assume a power law energy spectrum given by

$$F_o(E) = K E^{-\delta} \ . \tag{2.3.6}$$

In some limiting cases, we find that analytic solutions are possible. The analytic solutions are useful in many ways. They can give good quantitative estimates within their domain of applicability; they can be used to test the accuracy of the complex numerical results, and they provide a qualitative guide for more complicated cases outside the range of applicability.

A. Collision dominated solutions ($R \ll 1$)

For high densities and low values of the field strength and electron energy, the electron transport is dominated by Coulomb collisions and we can set $S = 0$. As shown by LP analytic solutions are then possible for small pitch angles. If $\alpha_o^2 \ll 1$, the injected electrons are strongly beamed along the field lines and we can set $\mu = (1 - \alpha^2/2)$ and $(1 - \mu^2) = \alpha^2$ in Equation (2.3.1). If we ignore the $O(\alpha^2)$ terms, and define a normalized collisional column depth τ_c and energy parameter η by

$$d\tau_c = C\, ds \ , \qquad d\eta = \beta^2 dE \tag{2.3.7}$$

then the flux at any depth is given by Equation (18) of LP:

$$F(E, \mu, \tau_c) = \left(\frac{\beta[E]}{\beta[E(\eta + \tau_c)]} \right)^2 \frac{2e^{-\alpha^2/\alpha_c^2}}{\alpha_c^2} F_o\left[E(\eta + \tau_c)\right] \ . \tag{2.3.8}$$

The pitch angle distribution is gaussian at all τ_c with a dispersion α_c given by

$$\alpha_c^2 = \alpha_o^2 + \zeta(E, \tau_c) \ , \tag{2.3.9}$$

where

$$\zeta(E, \tau_c) = 2\xi \ln \left[\frac{E(\eta + \tau_c)}{E(\eta)} \times \frac{2 + E(\eta)}{2 + E(\eta + \tau_c)} \right] \ . \tag{2.3.10}$$

From Equation (2.3.7) we have $\eta = E^2/(E+1)$ and

$$E(\eta) = \frac{\eta}{2} \left(1 + \sqrt{1 + \frac{4}{\eta}} \right) \ . \tag{2.3.11}$$

Thus $E(\eta) = E$ and $[\beta(E)]^2 = 1 - 1/(E+1)^2$ relates the velocity β to energy.

1) Non-relativistic limit. For non-relativistic particles $E \ll 1$ and $\eta \approx E^2$. For $\tau_c \ll 1$, which will be the case for these particles because they lose most of their energy by $\tau_c \sim \eta \sim E^2$,

$$E(\eta + \tau_c) = E\sqrt{1 + \tau_c/E^2}$$

and

$$\zeta(E, \tau_c) = \xi \ln(1 + \tau_c/E^2) \ . \tag{2.3.12}$$

As shown by LP this solution is a good approximation up to very large values of τ_c/E^2 and for injected pitch angle distributions with values of α_o^2 up to 0.40; much larger than expected considering the assumptions made.

2) Relativistic limit. For the extreme relativistic ($E \gg 1$) electrons we have $\eta \approx E$ and $E(\eta + \tau_c) = E + \tau_c$. The diffusion in pitch angle is small and according to Equations (2.3.9) and (2.3.10) the dispersion α_c^2 does not change appreciably with depth:

$$\alpha_c^2 = \alpha_o^2 + \frac{4\xi\tau_c}{E(E + \tau_c)} \ . \tag{2.3.13}$$

This implies an increase in dispersion with depth from α_o^2 to $\alpha_o^2 + 4\xi/E$ which, except for highly beamed injection with $\alpha_o^2 E \ll 1$, is a small effect.

Equation (2.3.13), however, often overestimates the dispersion in pitch angle. We obtained the approximate solution Equation (2.3.8) by setting $\mu = 1$ in front of the $\partial\Phi/\partial s$ term on the left hand side of Equation (2.3.1). This is reasonable for the non-relativistic case where the neglected term is of order α^2 and is insignificant in comparison with diffusion term, which is of order unity. As the electron energy increases the diffusion term becomes smaller and for sufficiently high energies ($E\alpha_o^2 \gtrsim 1$) it becomes comparable to or smaller than the $O(\alpha^2)$ term neglected. Thus for relativistic electrons we need a more accurate treatment of the $\mu(\partial\Phi/\partial\tau_c)$ term. As shown in Appendix A in this limit the diffusion term can be treated as a perturbation leading to the approximate solution $(A.4)$ for extreme relativistic electrons and all angles,

$$F(E,\mu,\tau_c) = \left(\frac{\beta(E)}{\beta(E+\tau_c/\mu)}\right)^2 F_o(E+\tau_c/\mu)G(\mu) \ , \tag{2.3.14}$$

and to a more accurate solution for intermediate and high energies but for the small pitch angle regime (Equation $A.14$);

$$F(E,\mu,\tau_c) = \left(\frac{\beta[E]}{\beta[E(\eta+\tau_c)]}\right)^2 \frac{2e^{-\alpha^2/\tilde{\alpha}_c^2}}{\alpha_c^2} F_o[E(\eta+\tau_c)] \ , \tag{2.3.15}$$

where α_c^2 is given by Equations (2.3.9) and (2.3.10) and the dispersion as a function of depth is

$$\tilde{\alpha}_c^2 = \frac{\alpha_c^2}{[1+\delta_e\tau_c\alpha_c^2/2(2+\eta+\tau_c)]} < \alpha_c^2 \ . \tag{2.3.16}$$

Here $\delta_e = d\ln F_o(E)/d\ln E|_{(E+\tau_c)}$ and is equal to the spectral index δ if F_o is a power law. Note that Equations (2.3.15) and (2.3.16) reduce to the non-relativistic limits of Equations (2.3.8) through (2.3.12) in the proper limit $\eta \sim \tau_c \ll 1$. Hence we may use this corrected solution for all energies.

Table 2.3 gives the values for $\tilde{\alpha}_c^2$ and α_c^2 along with α_*^2, the dispersion obtained from numerical solutions of Equation (2.3.1) including only the collision terms, but without the small pitch angle approximation, for four values of α_o^2 at $\tau_c/\eta =$

1. For $\alpha_o^2 = 0.04$ the diffusion effects are more important ($4/E\alpha_o^2 \approx 5$ and $\delta\alpha_o^2/2 \approx 0.10$). Consequently, the dispersion increases with depth, while for $\alpha_o^2 = 0.40$ the reverse is true ($4/E\alpha_o^2 \approx 0.5$ and $\delta\alpha_o^2/2 \approx 1.0$) and the dispersion decreases. In all cases except for isotropic injection ($\alpha_o^2 = 40$), $\tilde{\alpha}_c^2$ provides an excellent approximation to the dispersion α_*^2. For $\alpha_o^2 \gg 1$ it is obvious that α_c^2 is not a good approximation but $\tilde{\alpha}_c^2$ provides a reasonable approximation. A comparison of the pitch angle distribution from the above analytic approximation [Equations (2.3.15) and (2.3.16)] with that from the exact numerical solution for a power law injected flux given by Equations (2.3.5) and (2.3.6) with $\delta = 5$ and $\alpha_o^2 = 0.40$ shows that the agreement is excellent at small pitch angles, and reasonable at somewhat higher pitch angles. This comparison gives an indication of both the accuracy of our numerical code and of the usefulness of the small pitch angle approximation.

Table 2.3

A comparison of approximate widths with numerical results for 10.6 MeV electrons, with $\delta = 5$, $\alpha_o^2 = 0.04$, 0.10, 0.40, and 40, at depth $\tau_c = \eta$. The subscript $*$ denotes a numerical result.

$4/E\alpha_o^2$	$\delta\alpha_o^2/2$	α_o^2	$\alpha_c^2(\eta)$	$\tilde{\alpha}_c^2(\eta)$	$\alpha_*^2(\eta)$
5.00	0.10	0.04	0.13	0.12	0.11
2.00	0.25	0.10	0.20	0.16	0.17
0.50	1.00	0.40	0.50	0.31	0.31
0.005	100.0	40.0	40.10	0.81	0.61

3) Flux integrated over pitch angle. In certain problems, for example those with straight magnetic field lines, and processes with isotropic cross sections, knowledge of the pitch angle distribution is not necessary. We define total flux of

electrons of a given energy at a given depth to be $F_\mu(E,\tau_c) \equiv \int_{-1}^{+1} F(E,\mu,\tau_c)d\mu$ and integrate the solutions from Equation (2.3.15) to obtain

$$F_\mu(E,\tau_c) = \frac{\beta^2 F_o[E(\eta+\tau_c)]}{\beta^2[E(\eta+\tau_c)][1+\delta_e\tau_c\alpha_c^2/2(2+\eta+\tau_c)]} \ , \qquad (2.3.17)$$

which, as shown by LP, is independent of α_o^2 for non-relativistic electrons for all values of α_o^2. For relativistic electrons, F_μ does depend on α_o^2 due to the presence of α_c^2 in the correction to the dispersion in Equation (2.3.16).

The qualitative behavior of F_μ is similar for all energies. The flux is constant from $\tau_c = 0$ to $\tau_c \sim \eta$ and afterwards decreases with increasing depth. For a power law injected flux, $F_o(E) = KE^{-\delta}$, we find

$$F_\mu \approx \begin{cases} KE(E^2+\tau_c)^{-(\delta+1)/2} & \text{for } E \ll 1 \\ K(E+\tau_c)^{-\delta}/[1+\delta\tau_c\alpha_c^2/2(E+\tau_c)] & \text{for } E \gg 1 \end{cases} . \qquad (2.3.18)$$

For large values of $\tau_c \gg \eta$, the flux falls off as ${\tau_c}^{-(\delta+1)/2}$ for $E \ll 1$ and as $\tau_c^{-\delta}$ for $E \gg 1$.

4) Spatially integrated flux. For a magnetic field which is both uniform and straight a useful quantity is $F_\tau(E,\mu)$, the flux integrated over column depth, which is defined as $F_\tau(E,\mu) \equiv \int_0^\infty F(E,\mu,\tau_c)d\tau_c$. This quantity can be sufficient for the study of spatially unresolved sources. The zeroth order approximation, Equation (2.3.14), integrated over τ_c gives

$$F_\tau(E,\mu) = \mu G(\mu) \int_E^\infty (\beta')^2 F_o(E')dE' \ . \qquad (2.3.19)$$

We cannot obtain an analytic expression for F_τ by integrating Equations (2.3.8) or (15) due to the complex τ_c dependence of the dispersions α_c^2 and $\tilde{\alpha}_c^2$. It is much easier to integrate the original equation over $0 < \tau_c < \infty$. The integral of the source term Σ is equal to

$$\int_0^\infty \Sigma d\tau_c = cF_o(E,\mu) = c\beta^2\Phi_o(E,\mu) \qquad (2.3.20)$$

and the equation becomes

$$\beta^2\gamma^2\frac{\partial\Phi_\tau}{\partial E}+\xi\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\Phi_\tau}{\partial\mu}\right]-\mu\beta^4\gamma^2\Phi_o=0\ , \tag{2.3.21}$$

where $\Phi_\tau(E,\mu)\equiv\int_0^\infty\Phi(E,\mu,\tau_c)d\tau_c$.

For relativistic energies the pitch angle diffusion term is small and as above we can treat it as a perturbation. We expand Φ_τ in terms of $1/E$ and include the first order correction due to diffusion. The zeroth order solution for the flux is (since $F_\tau=c\Phi_\tau$ for $E\gg1$)

$$F_\tau(E,\mu)=\mu\int_E^\infty F_o(E',\mu)dE'\ , \tag{2.3.22}$$

and the first order solution is

$$\begin{aligned}F_\tau(E,\mu)=&\mu\int_E^\infty F_o(E',\mu)dE'\\&+\xi\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial}{\partial\mu}\left[\mu\int_E^\infty\frac{dE'}{E'^2}\int_{E'}^\infty F_o(E'',\mu)dE''\right]\right)\ ,\end{aligned} \tag{2.3.23}$$

For $F_o(E,\mu)=F_o(E)G(\mu)$ this gives

$$F_\tau(E,\mu)=\mu G(\mu)\int_E^\infty F_o(E')dE'\left(1+\frac{\xi}{\delta_{_T}E}\frac{1}{\mu G(\mu)}\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial}{\partial\mu}\left(\mu G(\mu)\right)\right]\right)\ , \tag{2.3.24}$$

where we have defined

$$\delta_{_T}^{-1}=\frac{E\int_E^\infty E'^{-2}dE'\int_{E'}^\infty F_o(E'')dE''}{\int_E^\infty F_o(E')dE'}\ , \tag{2.3.25}$$

which is a measure of the energy spectral index (for a power law F_o, $\delta_{_T}=\delta$).

For the small pitch angle regime with the injected gaussian distribution given in Equation (2.3.5), we find

$$F_\tau(E,\mu)=\left[\frac{2e^{-\alpha^2/\alpha_{_E}^2}}{\alpha_{_E}^2(1+\alpha_o^2/2)}\right]\int_E^\infty F_o(E')dE'\ , \tag{2.3.26}$$

where

$$\alpha_{_E}^2 = \frac{\alpha_o^2}{1+\alpha_o^2/2} + \frac{4\xi}{\delta_{_T} E} \ . \tag{2.3.27}$$

For the power law injected energy distribution $\delta_{_T} = \delta$ and because the dependence on energy of $\alpha_{_E}^2$ is small ($E \gg 1$), the spatially integrated flux should have a power law index equal to $\delta - 1$.

5) The total energy spectrum. Integration of F_μ over τ_c or F_τ over μ gives us the total spectrum $F_{_{\rm TOT}}(E)$ relevant for situations with isotropic processes and spatially unresolved observations. This solution is obtained by integrating Equation (2.3.19) over $d\mu$. Thus

$$F_{_{\rm TOT}}(E) = \int_{-1}^{+1} \mu G(\mu) d\mu \int_{_E}^{\infty} (\beta')^2 F_o(E') dE' \propto \begin{cases} E^{-(\delta-2)} & \text{for } E \ll 1 \\ E^{-(\delta-1)} & \text{for } E \gg 1 \end{cases} , \tag{2.3.28}$$

where the last relation is applicable fo a power law injected spectrum. In Section III we will compare these analytic results with numerical ones.

B. Synchrotron dominated case ($R \gg 1$)

In the opposite limit of high magnetic fields and particle energies synchrotron losses dominate. A general time dependent solution for the case with a uniform magnetic field and including synchrotron losses ($d\ln B/ds = C = 0$) is given in Appendix A. For the steady-state case with continuous injection the result is given by Equation ($A.27$). This solution is valid for electrons of all energies, but it takes a simple form for relativistic energies. The behavior of non-relativistic electrons is qualitatively similar to that of relativistic electrons.

1) Relativistic limit. The relativistic limit of Equation ($A.27$) is given by Equation ($A.33$), which for an injected flux of the form $F(E,\mu,0) = F_o(E)G(\mu)$ reduces to

$$F(E,\mu,s) = \frac{F_o[E/(1-\tau_s/\tau_{scr})]G(\mu)}{(1-\tau_s/\tau_{scr})^2} \ , \tag{2.3.29}$$

where we have defined a dimensionless depth $\tau_s = sS$ and

$$\tau_{scr}(E,\mu) = \left(\frac{\mu}{(1-\mu^2)}\right)\frac{1}{E} \ . \tag{2.3.30}$$

At a given pitch angle, $F \to 0$ at τ_{scr} which decreases with increasing energy so that higher energy particles are stripped from the beam at smaller depths. The depth s_{cr} corresponding to τ_{scr} also decreases with increasing magnetic field strength through the B^2 dependence of S.

At a given τ_s, the flux becomes zero at a critical pitch angle $\alpha_{cr} = \cos^{-1}(\mu_{cr})$ given by

$$\mu_{cr}(E,\tau_s) = \frac{1}{2\tau_s E}\left(\sqrt{1+4\tau_s^2E^2}-1\right) \ . \tag{2.3.31}$$

Note that μ_{cr} increases as τ_s increases, and approaches 1 as $\tau_s \to \infty$. Electrons with higher pitch angles are stripped from the beam as the depth increases.

In order to see the initial trend of the distribution it is instructive to consider the small pitch angle regime for $\tau_s \ll \tau_{scr} = (\alpha_o^2 E)^{-1}$. In this case, we let $\mu = 1-\alpha^2/2$ and $G(\mu) = 2\alpha_o^{-2}e^{-\alpha^2/\alpha_o^2}$ to obtain

$$F(E,\mu,s) = F_o(E)\frac{2e^{-\alpha^2/\alpha_s^2}}{\alpha_o^2} \ , \tag{2.3.32}$$

where (unless $\delta_e < 2$)

$$\alpha_s^2 = \alpha_o^2\left[1+(\delta_e-2)\alpha_o^2\tau_s E\right]^{-1} \ , \tag{2.3.33}$$

with $\delta_e = d\ln F_o(E)/d\ln E$. The dispersion α_s^2 decreases with increasing energy, depth and magnetic field. The effects of the pitch angle term and the first order correction to the extreme relativistic approximation will add terms of order $1/E$ in the square brackets.

2) Spatially integrated flux. The general expression for the flux integrated over depth is given by Equation $(A.34)$. In the relativistic limit we let $\beta \to 1$ and

$\beta' \to 1$ which gives

$$F_\tau(E,\mu) = \frac{\mu G(\mu)}{(1-\mu^2)E}\int_E^\infty F_o(E')dE'$$
$$= \frac{K\mu G(\mu)}{(1-\mu^2)E^{\delta+1}(\delta-1)}\ , \qquad (2.3.34)$$

where the second equality is for the power law injected flux. [Note that this expression may also be obtained by integration of Equation (2.3.29) over τ_s.] For a given pitch angle we have a spectral index of $(\delta+1)$ for this case.

3) Total energy spectrum. Integration of F_τ over pitch angle (or F_μ over depth) will give the total energy spectrum. However, if $\lim_{\mu\to 1} G(\mu) \neq 0$ (e.g. isotropic injection), the resulting expression diverges. This is because electrons with zero pitch angle never lose energy or change pitch angle; thus with a continuous injection there will be an infinite number of them from $0 < s < \infty$. This divergence disappears if as $\mu \to 1$, $G(\mu) \to (1-\mu^2)^\epsilon$, $\epsilon > 0$ and the total spectrum will be the same as that in Equation (2.3.34). The divergence will also be absent in the more realistic case of finite injection time or when collisions are included.

C. Synchrotron and Collisional losses

We need to consider both synchrotron and collisional losses when the ratio of these losses [R in Equation (2.3.4)] is near unity. For non-relativistic electrons $R \approx 1$ only when B is large. For relativistic electrons, however, synchrotron losses can be important for moderate values of B^2 if the density is low. There is no analytic solution for the general case including the synchrotron and collisional energy losses. Analytic solutions are possible for relativistic electrons because, as we have seen in sections A and B, Coulomb collisions and synchrotron radiation do not alter the pitch angles of relativistic electrons.

1) Relativistic limit. In this limit we can ignore the last two terms in Equation (2.3.1). The solution of this equation for uniform field and constant plasma

density (i.e., $E \gg 1$, $d\ln B/ds = 0$ and constant R_o) is given by Equation $(A.45)$ which reduces to

$$F(E,\mu,\tau_c) = F_o[E_*(E,\mu,\tau_c)]G(\mu)\frac{(1+E_*^2(E,\mu,\tau_c)/\varepsilon_c^2)}{(1+E^2/\varepsilon_c^2)} \ , \tag{2.3.35}$$

where

$$E_*(E,\mu,\tau_c) = E\left[\frac{1+(\varepsilon_c/E)\tan(\tau_c/\mu\varepsilon_c)}{1-(E/\varepsilon_c)\tan(\tau_c/\mu\varepsilon_c)}\right] \ , \tag{2.3.36}$$

and $\varepsilon_c^{-2} \equiv R_o(1-\mu^2)$.

Note that Equation (2.3.36) is valid only for $E\tan(\tau_c/\mu\varepsilon_c) \leq \varepsilon_c$. At a given pitch angle, $F \to 0$ at a depth given by

$$\tau'_{scr}(E,\mu) = \frac{E\tau_{scr}}{\varepsilon_c}\tan^{-1}\left(\frac{\varepsilon_c}{E}\right) \ , \tag{2.3.37}$$

where τ_{scr} is defined in Equation (2.3.30). As in the synchrotron-dominated case, for a given energy τ'_{scr} increases as μ increases, becoming infinite at $\mu = 1$. Then particles with high pitch angles are stripped away and the distribution narrows as depth increases. In the limit $\varepsilon_c \ll E$ (for $R_o \gg 1$) synchrotron losses dominate and the flux and the critical depth reduce to Equations (2.3.29) and (2.3.30) respectively. In the opposite limit ($R_o \ll 1$, $\varepsilon_c \gg E$) collisional losses dominate and Equation (2.3.35) reduces to Equation (2.3.14) as it must.

2) Spatially integrated flux. We cannot integrate the flux given in Equation (2.3.36) over pitch angle due to the complex μ dependence in E_* and ε_c but it is straightforward to integrate the flux over depth. We find

$$\begin{aligned} F_s(E,\mu) = \int_0^\infty ds F(E,\mu,s) &= \frac{\mu G(\mu)}{(1+E^2/\varepsilon_c^2)}\int_E^\infty F_o(E')dE' \\ &= \frac{\mu K G(\mu)}{(1+E^2/\varepsilon_c^2)(\delta-1)E^{\delta-1}} \ , \end{aligned} \tag{2.3.38}$$

where the last relation is for the power law injected flux. In the two limits $R_o \gg 1$ and $R_o \ll 1$ this equation reduces to the expressions in Equations (2.3.34) and (2.3.26) respectively.

3) Total energy spectrum. If the injected distribution is narrow (i.e., $\alpha_o^2 \ll 1$) we can integrate Equation (2.3.38) over pitch angle and obtain $F_{\rm TOT}$. We find

$$F_{\rm TOT}(E) = \frac{K e^{1/R_o E^2 \alpha_o^2}}{(\delta-1)\alpha_o^2 E^{\delta+1}} \left[\ln(R_o E^2 \alpha_o^2) - 0.577 + \sum_{k=1}^{\infty} \frac{(-R_o E^2 \alpha_o^2)^{-k}}{k k!} \right] , \tag{2.3.39}$$

which reduces to

$$F_{\rm TOT}(E) \approx \begin{cases} \dfrac{K}{(\delta-1)E^{\delta-1}} \left(1 - R_o E^2 \alpha_o^2\right) , & R_o E^2 \alpha_o^2 \ll 1 \\ \dfrac{K}{(\delta-1) R_o \alpha_o^2 E^{\delta+1}} \ln(R_o E^2 \alpha_o^2 / 1.78) , & R_o E^2 \alpha_o^2 \gg 1 \ . \end{cases} \tag{2.3.40}$$

Thus we have the expected spectral index for the collision dominated case at low energies and the index for the synchrotron dominated case (slightly modified by the logarithmic term) for high energies, provided that $R_o \alpha_o^2 \sim 1$. This modification is due to the fact that collisional losses dominate for electrons with very small pitch angles, $\alpha^2 \ll 1/R_o E^2$.

D. Non-uniform field ($d\ln B/ds \neq 0$)

Next we consider a non-uniform field for which $d\ln B/ds \neq 0$. We have no solution including collisional and/or synchrotron effects and a non-uniform field. [Ho (1986) has given numerical solutions of the equations of motion for the case including synchrotron losses and converging fields, but he has not solved the kinetic equation.] A solution for the case with $C = S = 0$ in Equation (2.3.1) was given in LP. In the absence of other effects $B/(1-\mu^2)$ is a constant for each particle which leads to the solution

$$F(E,\mu,s) = F_o(E) G\left(\sqrt{1 - (1-\mu^2) B_o / B(s)}\right) , \tag{2.3.41}$$

which for $G(\mu) = 2\alpha_o^{-2} e^{-(1-\mu^2)/\alpha_o^2}$, becomes

$$F(E,\mu,s) = \frac{2F_o(E)}{\alpha_o^2} \exp\left[\frac{-(1-\mu^2)B_o}{\alpha_o^2 B(s)}\right] . \tag{2.3.42}$$

At any point s, the distribution has a dispersion given by $\alpha_o^2 B(s)/B_o$; it is broadened by a factor of $B(s)/B_o$. This result is independent of energy and therefore the demonstration by LP of the accuracy of the numerical code is valid.

1) Integrated Fluxes. We can integrate the solution given in Equation (2.3.42) over pitch angle to obtain

$$F_\mu(E,s) = F_o(E)B(s)/B_o \ , \tag{2.3.43}$$

which increases with increasing magnetic field, since the distribution is broadened with no energy losses. It is clear that the energy dependence of the total flux $F_{\rm TOT}$ will be the same as that for F_o.

We also neglect the possible effects on the beam particles due to the generation of Langmuir waves in the ambient plasma. A necessary condition for the growth of Langmuir waves is the formation of a "gentle bump" distribution; i.e., one which has a positive value of $\partial f/\partial v_\parallel$. [See Emslie and Smith (1984) or Hamilton and Petrosian (1986) for details.] A distribution like this can result from collisions since the collisional terms are larger for lower energies (cf. Table 2.2). As we shall later see, the low energy particles are "stripped" from the beam as depth increases. Emslie and Smith (1984) showed that, if the beam can be approximated by a one-dimensional distribution, significant amounts of wave energy can be generated. This can have an effect on the beam distribution and can possibly account for the impulsive phase microwaves. We ignore this possibility for two reasons:

1.) The "bump" forms at low X-ray energies ($E \lesssim 20$ keV) and we are mostly concerned with higher energy particles. By the time a bump appears at high energies, the density ratio n_b/n_e is small enough so that the growth rate for Langmuir waves, which is proportional to n_b/n_e, is negligible.

2.) The one-dimensional calculations are not valid for beams which may be isotropic, and as we shall see in Section 2.3, regardless of the initial distribution, at low energies beams are nearly isotropic for depths on the order of the collisional energy loss scale. Collisional diffusion, which has $\dot{\mu}/\mu \propto \beta^{-3}$ is the dominant process at low energies; the collisional energy loss rate is a factor of β^2 smaller than the diffusion rate. [Compare $(\dot{\mu}/\mu)$ with the standard non-relativistic one-dimensional collisional frequency, $\nu(v) = \omega_p^4 \ln\Lambda/8\pi n_e v^3$.] Thus it is contradictory to assume that a beam is well collimated if a "bump" is formed at low energies; the process that forms the bump also makes the distribution isotropic. For example, using the approximate solution from Section 2.3 and the three-dimensional growth rate given in Melrose (1980), we find that growth is possible for $s > \delta E^2/C$ if we assume that the beam width α_o^2 remains small. However, at that depth the distribution is not narrow, as we shall see. It seems that a thorough three-dimensional calculation is necessary to deal with wave generation and that is beyond the scope of this work.

APPENDIX B

METHOD OF NUMERICAL SOLUTION

We start with Equation (2.3.1) from the text, the steady state equation including collisional and synchrotron losses, which we write as

$$\frac{\partial}{\partial E}(A_{_E}\,\Phi) = \beta^2\mu\frac{\partial\Phi}{\partial s} - \frac{\partial}{\partial\mu}(A_\mu\Phi) - D_\mu\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\Phi}{\partial\mu}\right] \ , \qquad (B.1)$$

where

$$A_{_E} = (C + S\beta^3\gamma^2(1-\mu^2)) \ , \qquad (B.2)$$

$$A_\mu = \frac{\beta^2(1-\mu^2)}{2}\frac{d\ln B}{ds} - \frac{S\beta\mu(1-\mu^2)}{\gamma} \ , \qquad (B.3)$$

and

$$D_\mu = \xi C/\beta^2\gamma^2 \ . \qquad (B.4)$$

To simplify matters, we write the right hand side of Equation $(B.1)$ as $\mathcal{F}[\Phi(E)]$, suppressing the μ and s in the arguments for now so that we have

$$\frac{\partial}{\partial E}A_{_E}(E)\Phi(E) = \mathcal{F}[\Phi(E)] \ . \qquad (B.5)$$

The method is essentially the same as that described in Leach (1984), differing only in that there are extra terms, and the final iteration is not the same as described in Leach (1984). [The other differences occur in the mechanics of the program, how various interpolations are handled, and the use of a different method of integration over the bremsstrahlung cross-section for radiation.] We may take advantage of the fact that there are no acceleration terms by stepping down in energy; i.e., given the solution at an energy E_1, we may obtain the solution at $E_o < E_1$ by writing

$$\frac{A_{_E}(E_1)\Phi(E_1) - A_{_E}(E_o)\Phi(E_o)}{(E_1 - E_o)} = \mathcal{F}[\Phi(E_{_{1/2}})] \qquad (B.6)$$

at the midpoint $E_{1/2}$, where

$$\mathcal{F}[\Phi(E_{1/2})] = \frac{1}{2}\left\{\mathcal{F}[\Phi(E_1)] + \mathcal{F}[\Phi(E_o)]\right\} \ . \tag{B.7}$$

Then

$$A_{E}(E_o)\Phi(E_o) + \frac{\Delta E}{2}\mathcal{F}[\Phi(E_o)] = A_{E}(E_1)\Phi(E_1) - \frac{\Delta E}{2}\mathcal{F}[\Phi(E_1)] \ , \tag{B.8}$$

where $\Delta E = E_1 - E_o$. Since we know the solution $\Phi(E_1)$, we know the entire right hand side of Equation $(B.8)$ and we write it as B_1.

Next we must deal with the derivatives over s and μ. We solve the equation on a grid which we will denote by s_j, μ_k, and we will write $\Phi_o(j,k) = \Phi(E_o, s_j, \mu_k)$ as the solution on each grid point. We define $\mathcal{F} = \eta_s + \zeta_\mu$, where

$$\eta_s(\Phi) = \mu\beta^2\frac{\partial\Phi}{\partial s} \tag{B.9}$$

and

$$\zeta_\mu(\Phi) = -\frac{\partial}{\partial\mu}(A_\mu\Phi) - D_\mu\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\Phi}{\partial\mu}\right] \ . \tag{B.10}$$

We multiply the whole equation by $2E_{1/2}\Delta s\Delta\mu$ and solve

$$2E_{1/2}\Delta s\Delta\mu\Phi_o(j,k) + E_{1/2}\Delta s\Delta\mu\Delta E\left[\eta_s + \zeta_\mu\right]\Phi_o(j,k) = 2E_{1/2}\Delta s\Delta\mu B_1 \ . \tag{B.11}$$

For $\mu > 0$, we write

$$\Delta s\eta_s(\Phi) = \mu\beta^2[\Phi_o(j,k) - \Phi_o(j-1,k)] \ , \tag{B.12}$$

and solve the equation at $s_{j-1/2}$, the midpoint between s_j and s_{j-1}. We have

$$E_{1/2}\Delta s\Delta\mu A_{E}[\Phi_o(j-1,k) + \Phi_o(j,k)] + \frac{E_{1/2}\Delta E\Delta s\Delta\mu}{2}\zeta_\mu[\Phi_o(j-1,k) + \Phi_o(j,k)]$$
$$+E_{1/2}\Delta E\mu\Delta\mu\beta^2[\Phi_o(j,k) - \Phi_o(j-1,k)] = 2E_{1/2}\Delta s\Delta\mu B_1(j-\tfrac{1}{2},k) \ , \tag{B.13}$$

for $\mu > 0$, and

$$\Delta s\eta_s(\Phi) = \mu\beta^2[\Phi_o(j+1,k) - \Phi_o(j,k)] \ , \tag{B.14}$$

and

$$E_{1/2}\Delta s\Delta\mu A_{E}[\Phi_o(j,k)+\Phi_o(j+1,k)]+\frac{E_{1/2}\Delta E\Delta s\Delta\mu}{2}\zeta_\mu[\Phi_o(j,k)+\Phi_o(j+1,k)]$$
$$+E_{1/2}\Delta E\mu\Delta\mu\beta^2[\Phi_o(j+1,k)-\Phi_o(j,k)]=2E_{1/2}\Delta s\Delta\mu B_1(j+\tfrac{1}{2},k)\ , \quad (B.15)$$

for $\mu<0$, where we have used $2\Phi_o(j\pm\frac{1}{2},k)=[\Phi_o(j\pm1,k)+\Phi_o(j,k)]$

Equation $(B.15)$ is solved by an iterative method. To find Φ_o^m for the mth iteration we assume that we have a solution $\Phi_o^{m-1}(j,k)$ for all j and k. Each iteration consists of a "sweep" down the loop and another sweep back up. We start by sweeping down; we inject $\Phi_o^m(j=1,k)$ and solve for Φ_o^m at each succeeding depth step. At a given j, then, we know $\Phi_o^m(j-1)$. For $\mu>0$ we have

$$E_{1/2}\Delta s\Delta\mu A_{E}\Phi_o^m(j,k)+\frac{E_{1/2}\Delta E\Delta s\Delta\mu}{2}\zeta_\mu\Phi_o^m(j,k)+E_{1/2}\Delta E\mu\Delta\mu\beta^2\Phi_o^m(j,k)$$
$$=2E_{1/2}\Delta s\Delta\mu B_1(j-\tfrac{1}{2},k)-E_{1/2}\Delta s\Delta\mu A_{E}\Phi_o^m(j-1,k)$$
$$-\frac{E_{1/2}\Delta E\Delta s\Delta\mu}{2}\zeta_\mu\Phi_o^m(j-1,k)+E_{1/2}\Delta E\mu\Delta\mu\beta^2\Phi_o^m(j-1,k)\ . \quad (B.16)$$

Since we are sweeping down, we must use the solution $\Phi_o^{m-1}(j+1,k)$ to find the solution for $\mu<0$. We have

$$E_{1/2}\Delta s\Delta\mu A_{E}\Phi_o^m(j,k)+\frac{E_{1/2}\Delta E\Delta s\Delta\mu}{2}\zeta_\mu\Phi_o^m(j,k)-E_{1/2}\Delta E\mu\Delta\mu\beta^2\Phi_o^m(j,k)$$
$$=2E_{1/2}\Delta s\Delta\mu B_1(j+\tfrac{1}{2},k)-E_{1/2}\Delta s\Delta\mu A_{E}\Phi_o^{m-1}(j+1,k)$$
$$-\frac{E_{1/2}\Delta E\Delta s\Delta\mu}{2}\zeta_\mu\Phi_o^{m-1}(j+1,k)-E_{1/2}\Delta E\mu\Delta\mu\beta^2\Phi_o^{m-1}(j+1,k)\ . \quad (B.17)$$

For the sweep up, the situation is reversed and we use $\Phi_o^m(j+1)$ for $\mu<0$ and $\Phi_o^{m-1}(j-1)$ for $\mu>0$. We now know the right hand sides of Equations $(B.16)$ and $(B.17)$, and are able to solve these equations using a matrix inversion method. The operator ζ_μ contains the diffusion term, with the 2nd derivative with respect to μ, and results in a tridiagonal matrix equation at each depth step after we write ζ_μ as a difference operator. Equations $(B.16)$ and $(B.17)$ then become

$$\Delta\mu\zeta_\mu\Phi_o(k)\equiv\Delta\mu Z_\mu=-\tfrac{1}{2}[A_\mu\Phi_o(k+1)-A_\mu\Phi_o(k-1)]$$

$$-D_\mu \left[\frac{(1-\mu^2_{k+1/2})(\Phi_o(k+1)-\Phi_o(k))}{\mu_{k+1}-\mu_k} - \frac{(1-\mu^2_{k-1/2})(\Phi_o(k)-\Phi_o(k-1))}{\mu_k-\mu_{k-1}} \right] , \qquad (B.18)$$

or

$$\Delta\mu Z_\mu \Phi_o(k) = \Phi_o(k+1)\left[-\frac{A_\mu}{2} - \frac{D_\mu(1-\mu^2_{k+1/2})}{\mu_{k+1}-\mu_k} \right] + \Phi_o(k)\left[\frac{D_\mu(1-\mu^2_{k+1/2})}{\mu_{k+1}-\mu_k} - \frac{D_\mu(1-\mu^2_{k-1/2})}{\mu_k-\mu_{k-1}} \right] + \Phi_o(k-1)\left[\frac{A_\mu}{2} - \frac{D_\mu(1-\mu^2_{k-1/2})}{\mu_k-\mu_{k-1}} \right] , \qquad (B.19)$$

where we have suppressed the j subscript.

In order to deal with the boundary conditions, and with the first iteration and the top of the energy grid, we start at energy E_{io} and assume $\Phi(E_{io+1}) = 0$. Also we start the first iteration with $\Phi^0_{io} = 0$ everywhere except at $s = 0$ (which we denote as $s(j = 0) = 0$ or just $j = 0$). So for the solution $\Phi^1_{io}(j = 1, k)$, Equation (B.16) for $\mu > 0$ becomes

$$E_{1/2}\Delta s\Delta\mu A_{_E}\,\Phi^1_{io}(1,k) + \frac{E_{1/2}\Delta E\Delta s\Delta\mu}{2} Z_\mu \Phi^1_{io}(1,k) + E_{1/2}\Delta E\mu\Delta\mu\beta^2\Phi^1_{io}(1,k) = -E_{1/2}\Delta s\Delta\mu A_{_E}\,\Phi^1_{io}(0,k) - \frac{E_{1/2}\Delta E\Delta s\Delta\mu}{2} Z_\mu \Phi^m_{io}(0,k) + E_{1/2}\Delta E\mu\Delta\mu\beta^2\Phi^m_{io}(0,k) \ . \qquad (B.20)$$

For the first sweep down, of course, $\Phi^1_{io}(\mu < 0)$ remains zero since $\Phi^0_{io}(j+1, k) = 0$. This part of Φ_{io} fills on the sweep up. Some care must be taken to insure that E_{io} is much larger than any energy in which we are interested, as it takes a number of steps down in energy to get accurate solutions. We use $E_{io} = 300$ MeV, which is large enough for accurate solutions at 20 MeV.