Basic Plasma Physics Principles

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Outline
Single particle orbits; drifts
Magnetic mirroring
MHD Equations
Force-free fields
Resistive Diffusion
The Vlasov equation; plasma waves

Single particle orbits
\[ F = q(E + v \times B) \]
Set \( E = 0 \) (for now)
\[ F = qv \times B \]
Since \( F \perp v \), no energy gain (\( F \cdot v = 0 \))
Particles orbit field line
\[ r = \frac{mv}{qB} \] (gyroradius)
\[ \omega = \frac{v}{r} = \frac{qB}{m} \] (gyrofrequency)

Motion in a Uniform Magnetic Field

Drifts
\[ F = q(E + v \times B) \]
Now let \( E \neq 0 \)
Relativistic transformation of \( E \) and \( B \) fields:
\[ E' = \gamma(E + (v/c) \times B) \]
\[ B' = \gamma(B - (v/c) \times E) \]
\[ E'^2 - B'^2 = E^2 - B^2 \]
If \( E < B \) (so that \( E^2 - B^2 < 0 \)), transform to frame in which \( E = 0 \):
\[ v = c \frac{E \times B}{B^2} \]
In this frame, we get simple gyromotion
So, in "lab" frame, we get gyro motion, plus a drift, at speed
\[ v_D = c \frac{E \times B}{B^2} \]

Drifts
\[ F = q(E + v \times B) \]

Exercise:
What if \( E > B \)?

E \times B drift

Drifts
\[ \nu_D = c \frac{(E + v \times B)}{B^2} \]
\( E \) is "equivalent electric field"
Examples:
1. \( E \) is actual electric field: \( \nu_D = c \frac{(E + v \times B)}{B^2} \) (independent of sign of \( q \))
2. Pressure gradient: \( qE = -\nabla p \): \( \nu_D = c \frac{\nabla p \times (E + v \times B)}{qB^2} \) (dependent on sign of \( q \))
3. Gravitational field: \( mg = qE \): \( \nu_D = c \frac{(mc/q) g \times (E + v \times B)}{B^2} \) (dependent on \( m \) and sign of \( q \))

Puzzle: in absence of magnetic field, particles subject to \( g \) accelerate at the same rate and in the same direction; particles subject to \( E \) accelerate in opposite directions at a rate which depends on their mass. \( Why \) is the exact opposite true when a \( B \) is present?
**Magnetic Mirroring**

Adiabatic invariants

Slow change of ambient parameters: Action $\oint dq$ (e.g. Energy/frequency) is conserved

Apply this to gyromotion: $E = - (1/2) mv^2 / B + eB$ and $R^2 = \frac{m_0^2}{4\pi \epsilon_0}$

Then as $B$ slowly changes, $mv^\perp$ increases and so, to conserve energy, $p$ must decrease. This can be expressed as a mirroring force

$F = - p \perp / \gamma_0$.

This force causes particles to be trapped in loops with high field strengths at the ends. Note that a magnetic compression also acts as a reflecting wall; this will help us understand particle acceleration later.

**Plasma physics in principle**

- Solve equations of motion with initial $\mathbf{E}$ and $\mathbf{B}$:
  \[ m_0v_0^2/c^2 = q(E + [\mathbf{v}_0 \times \mathbf{B}]) \]
- Then use the resulting $\mathbf{v}$ and $\mathbf{r}$ to get charge density $\rho(r)$ and current density $j(r)$
- Then obtain the self-consistent $\mathbf{E}$ and $\mathbf{B}$ through Maxwell’s equations:
  \[ \nabla \times \mathbf{E} = \rho, \quad \nabla \times \mathbf{B} = (4\pi/c) j + \mathbf{E}/c \]
  \[ \text{"Lather, rinse, repeat"} \]

**Complete set of MHD Equations**

- Replace $-10^{27}$ coupled equations of motion by “averaged” fluid equations
- Neglect displacement current (plasma responds very quickly to charge separation); then body force
  \[ \mathbf{F} = (1/c) \mathbf{j} \times \mathbf{B} = (1/4\pi) (\nabla \times \mathbf{B}) \times \mathbf{B} \]

**Force – Free Fields**

Solutions:

1. $\mathbf{B} = 0$ (trivial)
2. $\nabla \times \mathbf{B} = 0$ (current-free – “potential” field)
3. Linear case: $\nabla \times \mathbf{B} = \mathbf{0}$
4. Full case: $\nabla \times \mathbf{B} = \mathbf{0}$

Note that taking the divergence of $\nabla \times \mathbf{B} = 0$ gives $0 = \nabla \cdot \mathbf{B} + \alpha \mathbf{B}$, so that $\nabla \cdot \mathbf{B} = 0$, i.e. it is constant on a field line.

**Resistive Diffusion**

Consider the Maxwell equation

$\nabla \times \mathbf{E} - \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}$

together with Ohm’s law

$\mathbf{j} = - \sigma (\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \eta \nabla \times \mathbf{B}$

Combined, these give:

$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + (\eta/4\pi) \nabla \times (\nabla \times \mathbf{B})$, i.e.

$\mathbf{D} \times \mathbf{B} = \mathbf{j} - \mathbf{v} \times \mathbf{B}$

where $\mathbf{D} = \eta^2/\sigma \mathbf{B}$ is the resistive diffusion coefficient.
Resistive Diffusion

\[ \frac{\partial B}{\partial t} = \nabla \times (v \times B) + D \nabla^2 B \]

The ratio of the two terms on the RHS:

\[ \left| \nabla \times (v \times B) \right| / D \left| \nabla^2 B \right| \]

is known as the magnetic Reynolds number \( S \). For \( S \ll 1 \), the plasma is essentially diffusion-free, for \( S \gg 1 \) the dynamics are driven by resistive diffusion.

For a flare loop, \( V \approx V_A \approx 10^8 \text{ cm s}^{-1}, L \approx 10^9 \text{ cm} \) and \( \eta \approx 10^{-7} \text{ T}^{-3/2} \approx 10^{-17} \). This \( S \approx 10^{14} \), and the plasma should be almost perfectly frozen in.

The timescale for energy release should be of order \( L^2 / D \approx 4 \pi L^2 / \eta c^2 \) (this is of order the timescale for resistive decay of current in an inductor of inductance \( L/c^2 \) and resistance \( R = \eta L / L^2 = \eta / L \)). For solar values, this is \( 10^{15} \text{ s} \approx 10^7 \text{ years}! \)

Summary to Date

• Solar loops are big (they have a high inductance)
• Solar loops are good conductors
• Solar loops have a low ratio of gas to magnetic pressure \( \beta \)

So:

The plasma in solar loops is tied to the magnetic field, and the motion of this field determines the motion of the plasma trapped on it.

Also:

It is very difficult to release energy from such a high-conductivity, high-inductance system!

The Vlasov Equation

Phase-space Distribution Function

This is defined as the number of particles per unit volume of space per unit volume of velocity space:

At time \( t \), number of particles in elementary volume of space, with velocities in range \( v \to v + dv \):

\[ f(r,v,t) \text{ d}^3r \text{ d}^3v \]

\( f(r,v,t) \) has units \( \text{cm}^{-3} \text{(cm s}^{-1})^{-3} \)

The Boltzmann Equation

This equation expresses the fact that the net gain or loss of particles in phase space is due to collisional depletion:

\[ \frac{d f}{d t} + v \cdot \nabla f + a \cdot \nabla v f = \left( \frac{\partial f}{\partial t} \right) c \]

The Boltzmann equation takes into account the self-consistent evolution of the \( E \) and \( B \) fields through the appearance of the acceleration term \( a \).

The Electrostatic Vlasov Equation

Setting \( B = 0 \), we obtain, in one dimension for simplicity, with \( q = -e \) (electrons)

\[ \frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} - \left( \frac{eE}{m} \right) \frac{\partial f}{\partial v} = 0. \]

Perturb this around a uniform density, equilibrium \( (E = 0) \) state \( f_0 = n_0 \)

\[ \frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + \left( q/m \right) (E + (v/c) \times B) \frac{\partial f}{\partial v} = 0. \]

Also consider Poisson’s equation (\( \nabla \cdot E = 4 \pi \rho \)):

\[ \frac{\partial E_i}{\partial x} = 4 \pi n_0 \delta g_i \text{ d}v \]

Combining,

\[ \frac{\partial E}{\partial x} = -i(4 \pi n_e) (E_f \delta g_i \text{ d}v \text{ d}v) \text{ d}v \]

Simplifying and defining the plasma frequency through \( \omega_{pe}^2 = 4 \pi n_e e^2/m \),

\[ \omega - k v = -i(4 \pi n_e) \delta g_i \text{ d}v \text{ d}v \text{ d}v \]

This is the dispersion relation for electrostatic plasma waves.
The Electrostatic Vlasov Equation

Integrating by parts, we obtain an alternative form
\[1 - \left(\frac{\omega_{pe}^2}{\omega^2}\right) \int g_0 dv \left(1 - \frac{kv}{\omega}\right)^2 = 0.\]
For a cold plasma, \(g_0 = \delta(v),\) so that we obtain
\[1 - \left(\frac{\omega_{pe}^2}{\omega^2}\right) = 0,\]
\[\text{i.e., } \omega = \omega_{pe}.\]

The Electrostatic Vlasov Equation

For a warm plasma, we expand the denominator to get
\[1 - \left(\frac{\omega_{pe}^2}{\omega^2}\right) \int g_0 dv \left[1 + 2\frac{kv}{\omega} + 3\frac{k^2v^2}{\omega^2} + \ldots\right] = 0,\]
i.e.,
\[1 - \left(\frac{\omega_{pe}^2}{\omega^2}\right) \left[1 + 3\frac{k^2<v>^2}{\omega^2} + \ldots\right] = 0,\]
where \(<v>^2 = kBT/m\) is the average thermal speed. This gives the dispersion relation
\[\omega^2 = \omega_{pe}^2 + 3 \left(\frac{kBT}{m}\right) k^2,\]
(cf. \(\omega^2 = \omega_{pe}^2 + c^2k^2\) for EM waves).

Dispersion relations

Electrostatic waves in a warm plasma:
\[\omega^2 = \omega_{pe}^2 + 3 \left(\frac{kBT}{m}\right) k^2,\]
Ion-acoustic waves (includes motion of ions):
\[\omega = kcs; \quad cs = \left[\frac{kBT_e + kBT_i}{m_i}\right]^{1/2},\]
(note electrons effectively provide quasi-neutrality).
Upper hybrid waves (includes B):
\[\omega^2 = \omega_{pe}^2 + \Omega_e^2; \quad \Omega_e = eB/m_e.\]

Two-Stream Instability

Alfvén waves:
\[\omega^2 = kV_A^2 \left[1 + \left(V_A^2/c^2\right)\right],\]
Magnetoelectric waves:
\[\omega^2 - \omega^2 - k^2(c_s^2 + V_A^2) + c_s^2V_A^2k^2\cos^2\theta = 0,\]
(\(\theta = \text{angle of propagation to magnetic field}\)).

Two-Stream Instability

Distribution with two maxima (one at zero, one at the velocity of the “beam”) is susceptible to the two-stream instability. This generates a large amplitude of plasma waves and affects the energetics of the particles.

Summary

High energy solar physics is concerned with the physics of plasma, which is a highly interacting system of particles and waves.

"Plasma physics is complicated" (J.C. Brown & D.F. Smith, 1980)