1 The Analytic Form of the HESSI Point-Spread Function

The point-spread function is the response of a single point source in a map. The present analysis is for the fundamental, but higher harmonics and the effects of shadowing or grid nonuniformities could readily be included.
The instantaneous probability $\Pi_\alpha(r,\theta)$ of photon passage at grid angle α is proportional to $cos(2\pi(r/p)cos(\alpha-\theta)-\gamma(\alpha))$, where γ is the collimator phase at the spin axis. Multiplying this by the response $\Pi_\alpha(r_0,\theta_0)$ to a point source at (r_0,α_0) gives an instananeous PSF, which when integrated over the roll angles in one half rotation gives the full PSF:

$$P = \int_\delta^{\pi+\delta} \Pi_\alpha(r,\theta)\Pi_\alpha((r_0,\theta_0)d\alpha$$

$$= \int_\delta^{\pi+\delta} cos(2\pi r\ cos(\alpha-\theta)-\gamma(\alpha))\ cos(2\pi r_0\ cos(\alpha-\theta_0)-\gamma(\alpha))\ d\alpha$$

We make the assumption that γ is independent of roll angle, at least during one half rotation. Changing variables, $\beta=\alpha-\delta$ and simplifying the formulas with $k=2\pi r/p$ and $k_0=2\pi r_0/p$,

$$P = \int_0^\pi cos(k\ cos(\beta-\theta-\gamma))cos(k_0\ cos(\beta-\theta_0-\gamma))d\beta$$

Using the cosine sum rule permits us to show the γ dependence explicitly:

$$P = \tfrac{1}{2}P_1(k,k_0,\theta,\theta_0) + \tfrac{1}{2}cos2\gamma\ P_2(k,k_0,\theta,\theta_0) + \tfrac{1}{2}sin2\gamma\ P_3(k,k_0,\theta,\theta_0)$$

where the functions P_1, P_2 and P_3 are:

$$P_1 = \int_0^\pi cos(k\ cos(\beta-\theta)-cos(\beta-\theta_0))d\beta,$$

$$P_2 = \int_0^\pi cos(k\ cos(\beta-\theta)+cos(\beta-\theta_0))d\beta,$$

and

$$P_3 = \int_0^\pi sin(k\ cos(\beta-\theta)+cos(\beta-\theta_0))d\beta$$

All of these integrals can be reduced to the form $\int_0^\pi F(zcos(\beta - \psi)d\beta$, where F is cos or sin. Let z_1 and z_2 be defined by:

$$z_{1,2} = \{k_0^2 + k^2 \pm 2kk_0cos(\theta - \theta_0)\}^{1/2}$$

and $\psi_{1,2}$ are defined by:

$$sin(\psi_{1,2}) = (ksin(\theta) \pm k_0sin(\theta_0))/z_{1,2}$$

Then we have

$$P_1 = \int_0^\pi cos(z_1cos(\beta - \psi_1))d\beta$$
$$P_2 = \int_0^\pi cos(z_2cos(\beta - \psi_2))d\beta$$
$$P_3 = \int_0^\pi sin(z_1cos(\beta - \psi_1))d\beta$$

It is easy to show that P_1 is independent of ψ_1 and P_2 is independent of ψ_2 (This follows from the transformation $\beta' = \beta - \psi_1$, which allows us to evaluate $\partial P_1/\partial\psi_1 = [cos(z_1cos(\beta')]_{-\psi_1}^{\pi-\psi_1} = 0$. Similarly for P_2.)
Both P_1 and P_2 may be written in terms of Bessel functions:

$$P_1 = J_0(z_1) \qquad \text{and} \qquad P_2 = J_0(z_2)$$

but P_3 does not reduce to a Bessel function, and it depends explicitly on ψ_1. Because of the mirror asymmetry of the sine, P_3 is zero when $\psi_1 = 0$, and when $\psi_1 = \pi/2$,

$$P_3(z_1, \psi_1 = \pi/2) = \pi\mathbf{H_0}(z_1)$$

where $\mathbf{H_0}(z)$ is the Struve function, $\frac{1}{\pi}\int_0^{\pi/2} sin(zcos\beta)d\beta$.
Some manipulation of integrals shows that P_3 may be thought of as a "partial" Struve function:

$$P_3 = \int_0^{\psi_1} sin(zcos\beta)d\beta.$$

Since P_3 is obviously an odd periodic function of ψ_1, it may be expressed as a Fourier series with the leading terms:

$$P_3 = [a_1(z)sin\psi + a_3(z)sin(3\psi) + \cdots]$$

The Fourier coefficients vanish for even n, and for odd n they are:

$$a_n = \int_0^{2\pi} \int_0^{2\pi} sin[z_1 cos(\beta - \psi_1) sin(n\psi_1) d\beta d\psi_1$$

This may be reduced to a single integral over ψ by changing variables to $\beta' = \beta - \psi_1$, and integrating over β'.

$$a_n = \frac{2}{n} \int_0^{2\pi} sin(z\ cos\psi) cos(n\psi) d\psi$$

Using the integral definition of the Bessel function,

$$J_n(z) = \frac{i^{-n}}{2\pi} \int_0^{2\pi} e^{izcos\psi} cosn\psi d\psi,$$

the Fourier coefficients may be written:

$$a_n = \frac{1}{ni} \int_0^{2\pi} \{e^{izcos\psi} - e^{-izcos\psi}\} cosn\psi\ d\psi$$

$$= \frac{1}{ni}\{(2\pi i^n)(J_n(z) - J_n(-z))\} = \frac{2\pi i^{n-1}}{n} J_n(z)\{1 - (-1)^n\}$$

So the leading terms in the Fourier expansion of P_3 are:

$$P_3 = 4\pi\{J_1(z_1) sin\psi_1 - (1/3) J_3(z_1) sin(3\psi_1) + \cdots\}$$

When $\psi = \pi/2$ this must reduce to $\pi \mathbf{H_0}(z)$, and it does, because of the expansion of $\mathbf{H_0}(z)$ in terms of Bessel functions:

$$\mathbf{H_0}(z) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{J_{2k+1}(z)}{2k+1}$$

So finally, the HESSI PSF may be written:

$$P = \frac{1}{2}[J_0(z_1) + cos2\gamma J_0(z_2)] + 2\pi sin2\gamma\{J_1(z_1) sin\psi_1 - (1/3) J_3(z_1) sin3\psi_1 + \cdots\}$$