The Chi-squared Statistic in the Poisson Limit

In comparing a model e_i of the observed photon count rate n_i, it is common to employ the statistic:

$$S = \sum_{i=1}^{N} \frac{(n_i - e_i)^2}{e_i}.$$

In the limit of large values of n_i, S follows the χ^2 distribution, but it does not when the values of the countrate n_i are all ones and zeroes, or if many values of n_i are very small. Since we expect that many gamma-ray events observed by HESSI will have only a few photons, whose distribution follows Poisson statistics, we must consider the best way to find confidence limits of models.

Following Cash (1979), we adopt the statistic:

$$C = -2(n_i \ln(e_i/n_i) - e_i + n_i).$$

It is easily shown that $C \to S$ as $n_i \to \infty$. In practice, $C \approx S$ if $n_i > 10$, and like S, C vanishes if $e_i = n_i$.

For the case where there are p parameters and p-q constraints, one obtains confidence limits by letting all p parameters vary, finding the mimimum $(C_p)_{min}$, and by varying p-q parameters to find the minimum $(C_{p-q})_{min}$. Then the statistic

$$\Delta C = (C_{p-q})_{min} - (C_p)_{min}$$

obeys the χ^2 probability distribution with q degrees of freedom.

In particular, if one wants to find confidence limits for a particular pixel of a Maximum Entropy map, it is necessary to perform two minimizations:

1. Compute the quantity $(C_p)_{min}$ from the $\{e_i\}$ found by the Maximum Entropy search through parameter (pixel) space.
2. Select one of the pixels to test for goodness of fit, then determine $(C_{p-1})_{min}$ by searching for the minimum of C using MEM for all but that one pixel, whose flux is held constant.

Finally compute ΔC, and determine the confidence level from the χ^2 distribution with one degree of freedom. By Wilk's theorem, this is valid as long as the total number of photons is large ($>> 10$), even if individual count rates are all zeroes and ones.