Visibility of an Albedo Source

1. The Image Plane

In the image (x,y) plane, it is well known that a good approximation for the scattered radiation (the albedo patch) from a point-source primary flare is given by a Moffat function:

$$F(x,y) = Ah^3(h^2 + x^2 + y^2)^{-3/2}$$

where A is the flux of the albedo source, h is the height of the primary unit point source, and x,y are the sky plane coordinates relative to the primary source.

The above formula, being symmetric in x and y, is valid only when the flare is at Sun center, and requires modification for displacements from sun center. The most important effect of such a displacement is fore-shortening – the contours of constant albedo brightness become ellipses. The second important effect is that the centers of the primary and albedo sources no longer coincide. In the present analyis, we ignore this shift, and take the center of the (x,y) coordinate to be at the geometric center of the (elliptical) albedo patch.

Let the angles of the local vertical vector relative to the z (Sun-earth) direction be α and β. For a Sun-centered albedo patch, $\alpha = \beta = 0$, and their cosines both equal 1.

For a displacement in the x direction, $cos(\beta) = 1$ and $|cos(\alpha)| < 1$

The albedo patch contours are then given by

$$F(x,y) = Ah^3(h^2 + (x\ sec(\alpha))^2 + y^2)^{-3/2}$$

Note that the flux of the albedo patch is assumed to be unchanged, given our hypothesis of perfectly isotropic scattering.

For a displacement of the albedo patch from the x axis by a CCW angle ϕ, the x and y coordinates must be rotated:

$$\xi = x\ cos(\phi) - y\ sin(\phi), \quad \eta = y\ cos(\phi) + x\ sin(\phi)$$

So that

$$F(x,y) = Ah^3(h^2 + (\xi\ sec(\alpha))^2 + \eta^2)^{-3/2}$$

2. The Fourier Plane

To compute the visibility G(u,v) of the image in the Fourier (u,v) plane, we must perform the Fourier transform:

$$G(u,v) = \int_{\infty}^{\infty}\int_{\infty}^{\infty} e^{i2\pi(ux+vy)} F(x,y)\ dxdy$$

From tables of Hankel transforms it may be seen that Fourier transform of a symmetric Moffat function is an exponential, but to exploit this fact and use it in the current context, we must first do a rotation in the (u,v) plane and do a dilation in the u direction.

Since the quantity (ux+vy) is essentially a dot product of the (u.v) vector with the (x,y) vector,

$$ux + vy = u'\xi + v'\eta$$

where u' and v' are the rotated coordinates

$$u' = u\ cos(\phi) - v\ sin(\phi), \quad v' = v\ cos(\phi) + u\ sin(\phi)$$

(This may be verified explicitly using straightforward algebra.)
Then we can use the fact that rotation leaves the unit area unchanged ($dx\ dy = d\xi\ d\eta$), to get the form in the rotated plane:

$$G(u,v) = \int_{\infty}^{\infty}\int_{\infty}^{\infty} e^{i2\pi(u'\xi+v'\eta)} F(x,y)\ d\xi\ d\eta$$

where F is expressed in terms of (ξ, η).

Because there is a factor $sec(\alpha)$ multiplying ξ in the Moffat function, a further transformation must be made. Let $w = u'\ cos(\alpha)$ and $\xi' = \xi\ sec(\alpha)$. Then $d\xi = cos(\alpha)\ d\xi'$

Finally, we get a transform of a symmetric Moffat function:

$$G(u,v) = cos(\alpha)\ \int_{\infty}^{\infty}\int_{\infty}^{\infty} e^{i2\pi(w\ \ \xi'+v'\ \ \eta)} F_2\ d\xi'\ d\eta$$

where

$$F_2 = Ah^3(h^2 + (\xi')^2 + \eta^2)^{-3/2}$$

The transform of a symmetric, unit Moffat function is $exp(-kh)$ where $k = \sqrt{(}u^2 + v^2)$, so we obtain the result:

$$G(u,v) = A\ cos(\alpha) exp(-h\sqrt{(u'\ cos(\alpha))^2 + v'^2})$$