Using The Superposition Principle

By superposition, any visibility (or modulation profile) for an extended source can be represented as a sum over the responses to a set of point sources. (This is just applied Fourier analysis.) The visibility for a point source if flux A at (x,y) is:

$$\mathbf{V}_{pt} = Ae^{2\pi i(ux+vy)} \tag{1}$$

where u and v are the coordinates in the Fourier plane at angle ϕ for grids of angular pitch p

$$u = 2\pi cos(\phi)/p, \qquad v = 2\pi sin(\phi)/p \tag{2}$$

For an RMC like RHESSI, the fundamental component of the response to a point source is a similar sinusoid, with suitable modifications for the slowly varying grid transmission and slit-slat ratios:

$$C_{pt} = F_0 \ g(\phi) \ [1 + \rho(\phi) \ m(\phi) \ cos(ux + vy - \Phi)] \tag{3}$$

In this expression, F_0 is the flux of the point source, $g(\phi)$ is the grid transmission $\times$ livetime at angle ϕ, $m(\phi)$ is the maximum modulation amplitude, $\rho(\phi)$ is the relative amplitude, and Φ is the phase at map center.

An extended source with a brightness distribution $I(x, y)$ (normalized to unity) will produce a modulation profile given by:

$$C_{ext} = F_0 \int\int I(xy) \ \{ \ g(\phi) \ [1 \ + \ m(\phi) \ cos(ux + vy - \Phi)]\} \ dxdy \tag{4}$$

This is considerably simplified for Gaussian sources, round sources, or ”separable” sources.

Gaussian Sources

It is readily shown that for an elliptical Gaussian of the form

$$G(x', y') = \frac{1}{\sqrt{\pi ab}} e^{-(x'/a)^2-(y'/b)^2} \tag{5}$$

where the coordinates have be rotated and shifted via:

$$x' = (x-x_0) \ cos\alpha - (y-y_0) \ sin\alpha, \qquad y' = (x-x_0) \ sin\alpha + (y-y_0) \ cos\alpha \tag{6}$$

and

$$u' = u\ cos\alpha - v\ sin\alpha, \qquad v' = u\ sin\alpha + v\ cos\alpha \tag{7}$$

produces the RHESSI response:

$$C_{gau}(\phi) = F_0 g(\phi)\{1 + m(\phi)e^{-\pi^2(u'^2a^2+v'^2b^2)}cos(u'x_0 + v'y_0 - \Phi) \tag{8}$$

This shows that the response is the same as a point source at (x_0, y_0) with amplitude reduced by the factor:

$$\rho_{gau}(\phi) = e^{-\pi^2([kacos(\phi-\alpha)]^2+[kbsin(\phi-\alpha)]^2)} \tag{9}$$

General Round Sources

If the profile I(x,y) of the source is not necessarily Gaussian but is round (independent of θ), the relative amplitude is given by the integral

$$\rho = 2\pi \int I(r)\ J_0(kr)rdr, \quad \text{where} \quad 2\pi \int I(r)rdr = 1 \tag{10}$$

Profiles of this form that may be integrated analytically are:

(a) Lorentzians

$$I(r) = \frac{1}{2\pi}\frac{a}{(r^2 + a^2)^{3/2}}; \qquad \rho(k) = e^{-ka} \tag{11}$$

$$I(r) = \frac{1}{2\pi}\frac{2a^2}{(r^2 + a^2)^2}; \qquad \rho(k) = kK_1(ka) \tag{12}$$

$$I(r) = \frac{1}{2\pi}\frac{3a^3}{(r^2 + a^2)^{5/2}}; \qquad \rho(k) = (1 + ka)e^{-ka} \tag{13}$$

b) Exponential

$$I(r) = \frac{1}{2\pi a^2}e^{-r/a}; \qquad \rho(k) = \frac{1}{(1 + k^2a^2)^{3/2}} \tag{14}$$

(c) Gaussian

$$I(r) = \frac{1}{2\pi a^2}e^{-\frac{1}{2}r^2/a^2}; \qquad \rho(k) = e^{-\frac{1}{2}k^2a^2} \tag{15}$$

(d) Pillbox

$$I(r) = \frac{1}{2\pi a^2} \text{ if } r \leq a, \text{and } 0 \text{ if } r > a; \qquad \rho(k) = 2J_1(ka)/ka \tag{16}$$

In all cases as $a \to 0$ or $k \to 0$, $\rho(k) \to 1$, as expected.

Separable Profiles

By "separable" we mean that the radial and azimuthal dependences are uncorrelated:

$$I(r) = R(r)T(\theta) \tag{17}$$

This gives the following result for the relative amplitude:

$$\rho_{sep}(\phi) = Re \int_0^\infty \int_0^{2\pi} R(r)T(\theta)e^{ikrcos(\theta-\phi)}e^{-i\Phi} rdr\ d\theta \tag{18}$$

We can Fourier analyze $T(\theta)$

$$T(\theta) = \sum_{m=-\infty}^{\infty} \tau_m e^{im\theta} \tag{19}$$

Then doing the azimuthal integrals we get:

$$\rho_{sep}(\phi) = Re \sum_{m=-\infty}^{\infty} \tau_m e^{im\phi - i\Phi} \int_0^\infty R(r)J_m(kr)rdr \tag{20}$$

Exponential Profile

An example of a function R(r) which has explicit integrals for all m is:

$$R(r) = p^2 e^{-pr} \qquad \text{normalized to} \int_0^\infty rR(r)dr = 1 \tag{21}$$

From Gray and Matthews(1958),

$$H(k,p) = p^2 \int_0^\infty rJ_m(kr)e^{-pr}dr = \mu^2 \left(\frac{1-\mu}{1+\mu}\right)^{m/2} [m+\mu] \quad \text{where} \quad \mu = p/\sqrt{p^2+k^2} \tag{22}$$

The above equation is asymptotically correct for $k << 1$ and all $p > 0$.

$$\lim_{k\to 0} H(k,p) = (m+1)\left(\frac{k}{2p}\right)^m \tag{23}$$

Numerical integration using QROMB corroborates the expression for H(k,p) to several decimal places, and it agrees with the special cases m=0 and m=1 in Gradshteyn and Rhyzik (1994):

$$\int_0^\infty rJ_0(kr)e^{-pr}dr = p/(k^2+p^2)^{3/2} \tag{24}$$

$$\int_0^\infty rJ_1(kr)e^{-pr}dr = k/(k^2+p^2)^{3/2} \tag{25}$$

Gaussian Profile

Another example of a function R(r) which has explicit integrals for all m is:

$$R(r) = a^2 e^{-\frac{1}{2}(r/a)^2} \qquad \text{normalized to} \int_0^\infty rR(r)dr = 1 \tag{26}$$

From Abramowitz and Stegun (1966),

$$a^2 \int_0^\infty rJ_m(kr)e^{-\frac{1}{2}(r/a)^2}dr = \frac{\Gamma(m/2+1)}{2^{m/2+1}\Gamma(m+1)}(ka)^m M(m/2+1, m+1, -2k^2a^2) \tag{27}$$

where M is the confluent hypergeometric function.

Lorentzian Profile

A third example of a function R(r) which has explicit integrals is:

$$R(r) = \frac{1}{2(p-1)}\frac{a^2}{(1+(r/a)^2)^p} \qquad \text{normalized to} \int_0^\infty rR(r)dr = 1 \tag{28}$$

$$\int_0^\infty \frac{rJ_m(kr)}{(1+(r/a)^2)^p}dr = (k/2)^{p-1}a^{-m}/\Gamma(p)K_{1-p}(ka) \tag{29}$$

Using $K_{-1/2} = \sqrt{\pi/2ka}e^{-ka}$,

$$\int_0^\infty \frac{rJ_m(kr)}{(1+(r/a)^2)^p}dr = e^{-ka} \tag{30}$$