Using The Superposition Principle

By superposition, any visibility (or modulation profile) for an extended source can be represented as a sum over the responses to a set of point sources. (This is just applied Fourier analysis.) The visibility for a point source if flux A at \((x,y)\) is:

\[
V_{pt} = Ae^{2\pi i (ux + vy)}
\]  

where \(u\) and \(v\) are the coordinates in the Fourier plane at angle \(\phi\) for grids of angular pitch \(p\)

\[
u = \frac{2\pi \cos(\phi)}{p}, \quad v = \frac{2\pi \sin(\phi)}{p}
\]

For an RMC like RHESSI, the fundamental component of the response to a point source is a similar sinusoid, with suitable modifications for the slowly varying grid transmission and slit-slit ratios:

\[
C_{pt} = F_0 \, g(\phi) \left[ 1 + \rho(\phi) \, m(\phi) \, \cos(ux + vy - \Phi) \right]
\]

In this expression, \(F_0\) is the flux of the point source, \(g(\phi)\) is the grid transmission \(\times\) livetime at angle \(\phi\), \(m(\phi)\) is the maximum modulation amplitude, \(\rho(\phi)\) is the relative amplitude, and \(\Phi\) is the phase at map center.

An extended source with a brightness distribution \(I(x,y)\) (normalized to unity) will produce a modulation profile given by:

\[
C_{ext} = F_0 \int \int I(xy) \{ g(\phi) [1 + m(\phi) \cos(ux + vy - \Phi)] \} \, dx \, dy
\]

This is considerably simplified for Gaussian sources, round sources, or "separable" sources.

**Gaussian Sources**

It is readily shown that for an elliptical Gaussian of the form

\[
G(x',y') = \frac{1}{\sqrt{\pi ab}} e^{-\left(\frac{x'}{a}\right)^2 - \left(\frac{y'}{b}\right)^2}
\]

where the coordinates have be rotated and shifted via:

\[
x' = (x-x_0) \cos \alpha - (y-y_0) \sin \alpha, \quad y' = (x-x_0) \sin \alpha + (y-y_0) \cos \alpha
\]
and

\[ u' = u \cos \alpha - v \sin \alpha, \quad v' = u \sin \alpha + v \cos \alpha \]  

produces the RHESSI response:

\[ C_{gau}(\phi) = F_{0g}(\phi) \left[ 1 + m(\phi) e^{-\pi^2(u'^2a^2 + v'^2b^2)} \cos (u'x_0 + v'y_0 - \Phi) \right] \]  

This shows that the response is the same as a point source at \((x_0, y_0)\) with amplitude reduced by the factor:

\[ \rho_{gau}(\phi) = e^{-\pi^2([k\cos(\phi-\alpha)]^2 + [k\sin(\phi-\alpha)]^2)} \]  

**General Round Sources**

If the profile \(I(x,y)\) of the source is not necessarily Gaussian but is round (independent of \(\theta\)), the relative amplitude is given by the integral

\[ \rho = 2\pi \int I(r) J_0(kr) r dr, \quad \text{where} \quad 2\pi \int I(r) r dr = 1 \]  

Profiles of this form that may be integrated analytically are:

(a) Lorentzians

\[ I(r) = \frac{1}{2\pi} \frac{a}{(r^2 + a^2)^{3/2}}, \quad \rho(k) = e^{-ka} \]  

(b) Exponential

\[ I(r) = \frac{1}{2\pi a^2} e^{-r/a}, \quad \rho(k) = \frac{1}{(1 + k^2 a^2)^{3/2}} \]  

(c) Gaussian

\[ I(r) = \frac{1}{2\pi a^2} e^{-\frac{1}{2}r^2/a^2}, \quad \rho(k) = e^{-\frac{1}{2}k^2 a^2} \]  

(d) Pillbox

\[ I(r) = \begin{cases} \frac{1}{2\pi a^2} & \text{if } r \leq a, \text{and } 0 \text{ if } r > a; \\ \rho(k) = 2J_1(ka)/ka \end{cases} \]  

In all cases as \(a \to 0\) or \(k \to 0\), \(\rho(k) \to 1\), as expected.
Separable Profiles

By "separable" we mean that the radial and azimuthal dependences are uncorrelated:

\[ I(r) = R(r)T(\theta) \]  

(17)

This gives the following result for the relative amplitude:

\[ \rho_{sep}(\phi) = \text{Re} \int_0^\infty \int_0^{2\pi} R(r)T(\theta)e^{i\phi}rdr d\theta \]  

(18)

We can Fourier analyze \( T(\theta) \)

\[ T(\theta) = \sum_{m=-\infty}^{\infty} \tau_me^{im\theta} \]  

(19)

Then doing the azimuthal integrals we get:

\[ \rho_{sep}(\phi) = \text{Re} \sum_{m=-\infty}^{\infty} \tau_me^{im\phi-i\Phi} \int_0^\infty R(r)J_m(kr)rdr \]

(20)

Exponential Profile

An example of a function \( R(r) \) which has explicit integrals for all \( m \) is:

\[ R(r) = p^2e^{-pr} \quad \text{normalized to} \int_0^\infty rR(r)dr = 1 \]

(21)

From Gray and Matthews(1958),

\[ H(k, p) = p^2 \int_0^\infty rJ_m(kr)e^{-pr}dr = \mu^2 \left( \frac{1 - \mu}{1 + \mu} \right)^{m/2} [m+\mu] \quad \text{where} \quad \mu = p/\sqrt{p^2 + k^2} \]

(22)

The above equation is asymptotically correct for \( k << 1 \) and all \( p > 0 \).

\[ \lim_{k \to 0} H(k, p) = (m + 1) \left( \frac{k}{2p} \right)^m \]

(23)

Numerical integration using QROMB corroborates the expression for \( H(k,p) \) to several decimal places, and it agreees with the special cases \( m=0 \) and \( m=1 \) in Gradshteyn and Rhyzik (1994):

\[ \int_0^\infty rJ_0(kr)e^{-pr}dr = p/(k^2 + p^2)^{3/2} \]

(24)
\[ \int_0^\infty r J_1(kr)e^{-pr} dr = \frac{k}{(k^2 + p^2)^{3/2}} \]  

(Gaussian Profile)

Another example of a function \( R(r) \) which has explicit integrals for all \( m \) is:

\[ R(r) = a^2 e^{-\frac{k}{2}(r/a)^2} \] normalized to \( \int_0^\infty rR(r)dr = 1 \)

From Abramowitz and Stegun (1966),

\[ a^2 \int_0^\infty rJ_m(kr)e^{-\frac{k}{2}(r/a)^2} dr = \frac{\Gamma(m/2 + 1)}{2^{m/2 + 1}\Gamma(m + 1)}(ka)^m M(m/2+1, m+1, -2k^2a^2) \]

(27)

where \( M \) is the confluent hypergeometric function.

(Lorentzian Profile)

A third example of a function \( R(r) \) which has explicit integrals is:

\[ R(r) = \frac{1}{2(p - 1)} \frac{a^2}{(1 + (r/a)^2)^p} \] normalized to \( \int_0^\infty rR(r)dr = 1 \)

(28)

\[ \int_0^\infty \frac{rJ_m(kr)}{(1 + (r/a)^2)^p} dr = \frac{(k/2)^{p-1}a^{-m}/\Gamma(p)K_{1-p}(ka)} \]

(29)

Using \( K_{-1/2} = \sqrt{\pi/2}ke^{-ka} \),

\[ \int_0^\infty \frac{rJ_m(kr)}{(1 + (r/a)^2)^p} dr = e^{-ka} \]

(30)